Ladder Topologies for Massive 3-Loop Operator Matrix Elements

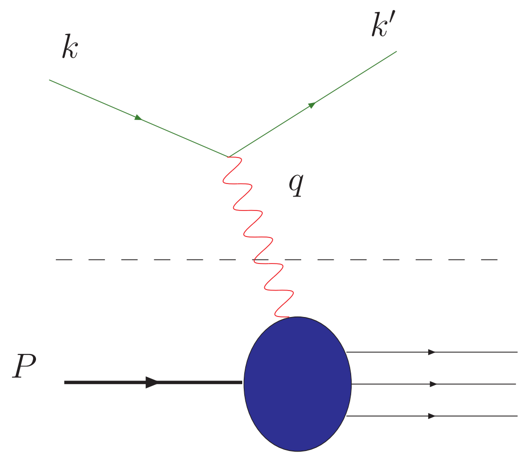
Alexander Hasselhuhn, DESY

[in collaboration with J. Blümlein (DESY), S. Klein (RWTH) and C. Schneider (JKU Linz)]

- Introduction
- Status of Heavy Flavor Contributions to DIS Structure Functions
- Computation Method of Heavy Flavor Wilson Coefficients
- Calculation of 3-Loop Ladder Graphs for Massive OMEs
- Automation
- Summary and Outlook

1. Introduction

Unpolarized Deep–Inelastic Scattering (DIS):



$\longrightarrow \quad L_{\mu\nu}$

$\longrightarrow \quad W_{\mu\nu}$

$$Q^2 := -q^2, \quad x := \frac{Q^2}{2P.q}$$ Bjorken–x

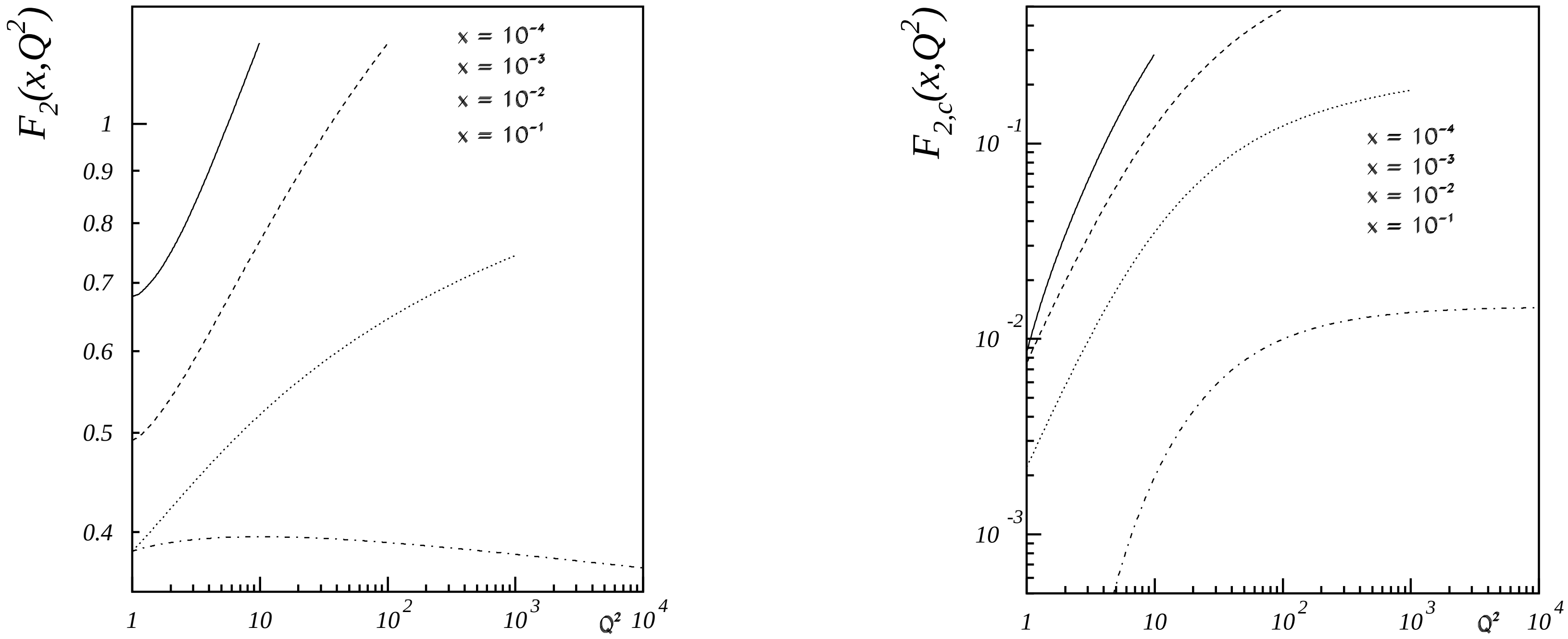
$$\frac{d\sigma}{dQ^2\, dx} \sim W_{\mu\nu} L^{\mu\nu}$$

$$W_{\mu\nu}(q,P,s) = \frac{1}{4\pi} \int d^4\xi \exp(iq\xi) \langle P,s \mid [J_\mu^{em}(\xi), J_\nu^{em}(0)] \mid P,s \rangle$$

$$= \frac{1}{2x}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) F_L(x,Q^2) + \frac{2x}{Q^2}\left(P_\mu P_\nu + \frac{q_\mu P_\nu + q_\nu P_\mu}{2x} - \frac{Q^2}{4x^2} g_{\mu\nu}\right) F_2(x,Q^2) \ .$$

Structure Functions: $F_{2,L}$

contain light and heavy quark contributions.



LO charm contributions: PDFs from [Alekhin, Melnikov, Petriello, 2006.]

→ different scaling violations,

→ massive contributions at lower values of x are of order 20%-35%.

Hence for the prediction of cross sections at the LHC the precise knowledge of all PDFs is needed.

2. Status of Heavy Flavor Contributions to DIS Structure Functions

Leading Order: [Witten, 1976; Babcock, Sivers, 1978;Shifman, Vainshtein, Zakharov, 1978; Leveille, Weiler, 1979; Glück, Reya, 1979; Glück, Hoffmann, Reya, 1982.]

Next-to-Leading Order : [Laenen, Riemersma, Smith, van Neerven, 1993, 1995]

asymptotic: [Buza, Matiounine, Smith, Migneron, van Neerven, 1996] via IBP

$(Q^2 \gg m^2)$ [Bierenbaum, Blümlein, Klein, 2007] via ${}_pF_q$'s, more compact results

NLO fast Mellin space implementation: [Alekhin, Blümlein 2003]

NNLO, $Q^2 \gg m^2$: contribs. to F_L for all N: [Blümlein, De Freitas, van Neerven, Klein 2006]

contributions to F_2 $(N = 2...10(14))$: [Bierenbaum, Blümlein, Klein 2009]

contributions to transversity $(N = 1...13)$: [Blümlein, Klein, Tödtli 2009]

all $O(\alpha_s^3) \times \ln^k \left(\frac{Q^2}{m^2}\right)$ terms for massive OMEs for general N: [Bierenbaum, Blümlein, Klein 2010]

contributions to $O(N_f T_f^2 C_{A,F})$ for all N:(→Talk by F. Wißbrock) [Ablinger, Blümlein, Klein, Schneider, Wißbrock 2010]

Goal: Calculate the 3-loop massive Wilson coefficients for $F_2(x, Q^2)$ in the region $Q^2 \gtrsim 10m^2$ for general values of N.

3. Computation Method of Heavy Flavor Wilson Coefficients

At leading twist the structure functions factorize in terms of a Mellin convolution

$$F_{(2,L)}(x,Q^2) = \sum_j \underbrace{\mathbb{C}_{j,(2,L)}\left(x, \frac{Q^2}{\mu^2}, \frac{m^2}{\mu^2}\right)}_{perturbative} \otimes \underbrace{f_j(x,\mu^2)}_{nonpert.}$$

into (pert.) Wilson coefficients and (nonpert.) parton distribution functions (PDFs).
$\otimes$ denotes the Mellin convolution

$$f(x) \otimes g(x) \equiv \int_0^1 dy \int_0^1 dz \, \delta(x - yz) f(y) g(z) \ .$$

The subsequent calculations are performed in Mellin space, where $\otimes$ reduces to a multiplication, due to the Mellin transformation

$$\hat{f}(N) := \int_0^1 dx \, x^{N-1} f(x) \ .$$

Wilson coefficients:

$$\mathbb{C}_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=C_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)+H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)\ .$$

At $Q^2 \gg m^2$ the heavy flavor part

$$H_{j,(2,L)}\left(N,\frac{Q^2}{\mu^2},\frac{m^2}{\mu^2}\right)=\sum_i C_{i,(2,L)}\left(N,\frac{Q^2}{\mu^2}\right)A_{ij}\left(\frac{m^2}{\mu^2},N\right)$$

[Buza, Matiounine, Smith, van Neerven 1995]

factorizes into the light flavor Wilson coefficients C and the massive operator matrix elements (OMEs) of local operators O_i between partonic states j

$$A_{ij}\left(\frac{m^2}{\mu^2},N\right)=\langle j \mid O_i \mid j\rangle\ .$$

→ additional Feynman rules with local operator insertions for partonic matrix elements.

The unpolarized light flavor Wilson coefficients are known up to NNLO

[Moch, Vermaseren, Vogt, 2005].

For $F_2(x,Q^2)$: at $Q^2 \gtrsim 10m^2$ the asymptotic representation holds at the 1% level.

Renormalization

The unrenormalized OMEs read:

$$\hat{A}_{ij} = \delta_{ij} + \sum_{k=1}^{\infty} \hat{a}_s^k \sum_{l=0}^{k} \frac{\hat{a}_{kl}}{\varepsilon^{k-l}} \ .$$

The renormalization requires the following steps, cf. [Bierenbaum, Blümlein, Klein 2009]:

- mass renormalization,
- charge renormalization,
- UV-divergence are absorbed into Z

$$\begin{aligned} O_q^{\rm NS} &= Z^{\rm NS} \hat{O}_q^{\rm NS} \\ O_i^{\rm S} &= Z_{ij}^{\rm S} \hat{O}_j^{\rm S} \ , \quad i,j = q,g \ , \end{aligned}$$

- collinear divergences in the graphs are factorized $\rightarrow \Gamma \neq Z$.

Thus the renormalized OME can be expressed as

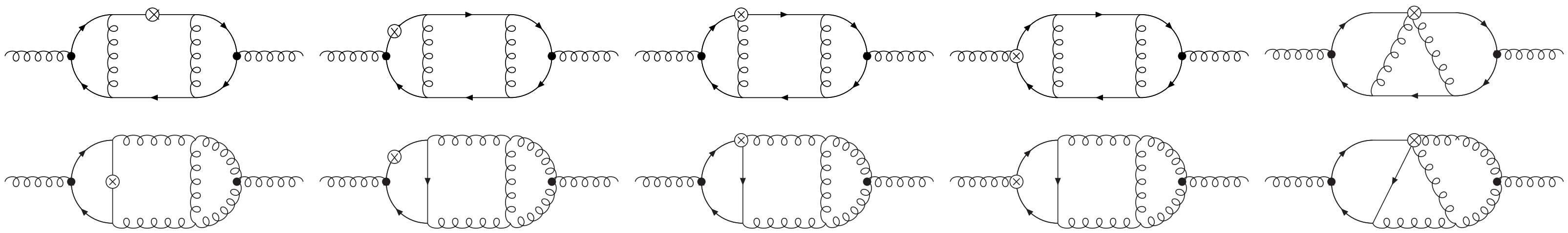
$$A_{ij}\left(N, \frac{m^2}{\mu^2}\right) = \delta_{ij} + \sum_{k=1}^{\infty} a_s^k \sum_{l=0}^{k} \ln^{k-l}\left(\frac{m^2}{\mu^2}\right) a_{kl}(N) \ .$$

4. Calculation of 3-Loop ladder graphs for massive OMEs

In the following the most complicated parts of three loop contributions to the massive OME A_{Qg} with ladder topology are calculated.

The steps of the calculation are:

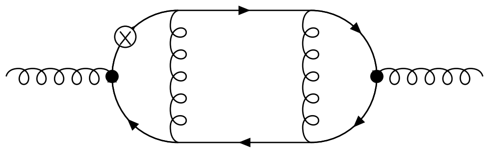
- Generate the momentum integral using the Feynman rules encoded in QGRAF + operator insertions. [Bierenbaum, Blümlein, Klein 2009] [Nogueira 1993]



.

- Apply the Feynman parametrization.
- Identify the characteristic hypergeometric structure: ${}_pF_q$'s, Appell's functions and generalizations thereof.
- After representing the integrals as sums over hypergeometric functions, the expansion in ε leads to nested sums over hypergeomtric terms equipped with harmonic sums.
- Perform these sums using modern symbolic summation methods. The results are expressed in terms of rational expressions, harmonic sums and generalizations thereof.

Example 1: Feynman parametrization



The Feynman rules provide us with the following integral ($\hat{d}k \equiv \frac{d^D k}{(2\pi)^D}$ and $D = 4 + \epsilon$):

$$I_{2a} := K_{2a} \iiint \frac{\hat{d}k\ \hat{d}r\ \hat{d}s \qquad (\Delta.k)^{N-1}}{((k-p)^2-m^2)((r-p)^2-m^2)((s-p)^2-m^2)(s^2-m^2)(r^2-m^2)(k^2-m^2)(k-r)^2(s-r)^2}\ .$$

Apply Feynman parametrization proceeding from outer to inner loops

$$I_{2a} = \tilde{K}_{2a} S_\varepsilon^3 e^{-\gamma_E \varepsilon/2}\, \Gamma\left(2 - \frac{3\varepsilon}{2}\right) \int_{[0,1]^7} dx\, dz\, du\, dw\, da\, ds\, dt\ z^{\frac{\varepsilon}{2}-1}(1-z)^{\frac{\varepsilon}{2}}(1-u) w^{\frac{\varepsilon}{2}-1}(1-w)^{\frac{\varepsilon}{2}} \times$$
$$\times s^{-\frac{\varepsilon}{2}} t^{-\frac{\varepsilon}{2}} \theta(1-s-t)(1-s-t)\left(1 - s\frac{z-1}{z} - t\frac{w-1}{w}\right)^{-2+3\varepsilon/2} \times$$
$$\times (u(1-w) + wa(1-s-t) + wsx + wtu)^{N-1}\ .$$

The operator insertion contributes as an integer power of a polynomial linear in each Feynman parameter. Therefore the binomial theorem may be applied.

Example 1: Hypergeometric Series

The topology of massive lines leads to characteristic terms of an integral representation of Appell's function F_1

$$F_1\left[a; b, b'; c; X, Y\right] = \int_0^1 ds \int_0^1 dt \, \frac{\theta(1-s-t)\, s^{b-1}\, t^{b'-1}\, (1-s-t)^{c-b-b'-1}}{(1-sX-tY)^a} \; .$$

The form of the arguments of F_1 is independent of the operator insertion.

$$F_1\left[a; b, b'; c; \frac{w-1}{w}, \frac{z-1}{z}\right]$$

Its standard series representation does not converge for $w, z \in [0,1]$. Therefore an analytic continuation formula is needed:

$$F_1\left[a; b, b'; c; \frac{w-1}{w}, \frac{z-1}{z}\right] = w^b z^{b'} F_1\left[c-a; b, b'; c; (1-w), (1-z)\right] \; ,$$

which can be represented via:

$$F_1\left[c-a; b, b'; c; (1-w), (1-z)\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(c-a)_{m+n}(b)_m(b')_n}{m!\; n!\; (c)_{m+n}} (1-w)^m (1-z)^n \; .$$

The Pochhammer symbol is defined as

$$(a)_m \equiv a(a+1)...(a+m-1) \; .$$

Example 1: Hypergeometric representation

- The application of the binomial theorem leads to an integral for F_1.
- One applies the series representation for F_1 and performs beta-type integrals.

$\longrightarrow$ binomial sums over a hypergeometric series:

$$
\begin{aligned}
I_{2a} &= \tilde{K}_{2a} S_\varepsilon^3 e^{-\gamma_E 3\varepsilon/2} \Gamma\left(2 - \frac{3}{2}\varepsilon\right) \frac{1}{(N+1)(N+2)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=2}^{N+2} \binom{N+2}{l} \sum_{j=2}^{l} \binom{l}{j} \Biggl\{ \\
&\times \sum_{k=1}^{j} \binom{j}{k} \sum_{r=0}^{l-k} \binom{l-k}{r} (-1)^{l+j+k+r} B\left(k, m+1+\frac{\varepsilon}{2}\right) \\
&\times \Gamma\left[\begin{array}{c} k+r+j+m+n+\frac{\varepsilon}{2} \\ m+1, n+1, k+r+\frac{\varepsilon}{2} \end{array}\right] \frac{B\left(k+m-\frac{\varepsilon}{2}, r+1+n-\frac{\varepsilon}{2}\right) B\left(r+l-1, n+1+\frac{\varepsilon}{2}\right)}{(k+r+1+m+n-\varepsilon)(N+3-j)} \\
&+ \sum_{r=0}^{l-j} \binom{l-j}{r} (-1)^{l+j+r} B\left(j, m+1+\frac{\varepsilon}{2}\right) \\
&\times \Gamma\left[\begin{array}{c} j+r+m+n+\frac{\varepsilon}{2} \\ m+1, n+1, j+r+\frac{\varepsilon}{2} \end{array}\right] \frac{B\left(j+m-\frac{\varepsilon}{2}, r+1+n-\frac{\varepsilon}{2}\right) B\left(r+l-1, n+1-\frac{\varepsilon}{2}\right)}{(j+r+1+m+n-\varepsilon)(N+3-j)} \Biggr\} .
\end{aligned}
$$

This expression is finite as $\varepsilon \to 0$.

Example 1: Symbolic Summation

Sums over hypergeometric expressions equipped with harmonic sums are solved using the package Sigma [C. Schneider 2007] applying modern symbolic summation techniques.
The result can be written in terms of rational expressions, harmonic sums and generalizations thereof in N:

$$\begin{aligned} I_{2a} &= \frac{\tilde{K}_{2a} S_\varepsilon^3}{(N+1)(N+2)(N+3)} \Bigg\{ \frac{1}{6} S_1^3 + \frac{N^2+12N+16}{2(N+1)(N+2)} S_1^2 + \frac{4(2N+3)}{(N+1)^2(N+2)} S_1 \\ &+ \frac{8(2N+3)}{(N+1)^3(N+2)} + 2\left[-2^{N+3} + 3 - (-1)^N\right]\zeta_3 - (-1)^N S_{-3} + \left[\frac{3N^2+40N+56}{2(N+1)(N+2)} - 2S_1\right] S_2 \qquad (1) \\ &- \frac{3N+17}{3} S_3 - 2(-1)^N S_{-2,1} - (N+3) S_{2,1} + 2^{N+4} S_{1,2}\left(\frac{1}{2}, 1\right) + 2^{N+3} S_{1,1,1}\left(\frac{1}{2}, 1, 1\right) \Bigg\} + O(\varepsilon) \,. \end{aligned}$$

Where we used the shorthand for harmonic sums and generalizations defined resp.:

$$S_{b,\vec{a}}(N) = \sum_{k=1}^{N} \frac{\mathrm{sign}(b)^k}{k^{|b|}} S_{\vec{a}}(k) \quad \to S_{b,\vec{a}} \,, \quad b, a_i \in \mathbb{Z}\backslash\{0\}$$

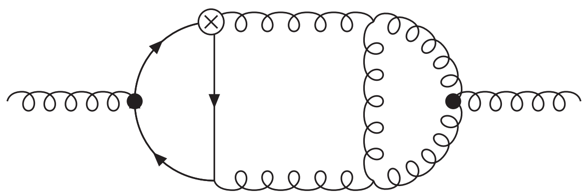
[Blümlein, Kurth 1998]
[Vermaseren 1998]

$$S_{b,\vec{a}}(\eta, \vec{\xi}; N) = \sum_{k=1}^{N} \frac{\eta^k}{k^b} S_{\vec{a}}(\vec{\xi}, k) \quad \to S_{b,\vec{a}}(\eta, \vec{\xi}) \,, \quad b, a_i \in \mathbb{N}^+, \ \eta, \xi_i \in \mathbb{R}$$

[Moch, Uwer, Weinzierl 2002]
[Ablinger, Blümlein, Schneider 2010] .

In (1) the powers 2^N lead to divergences as $N \to \infty$ and are therefore expected to cancel in the full expression.

Example 2: Hypergeometric representation



The momentum integral reads:

$$I_9 = K_9 \iiint \hat{d}k\hat{d}r\hat{d}s \sum_{j=0}^{N} \frac{(\Delta.k)^j(\Delta.k-\Delta.r)^{N-j}}{((k-p)^2-m^2)(k-r)^2(k-s)^2 s^2 r^2 (k^2-m^2)((k-r)^2-m^2)(s-r)^2} \;.$$

The Feynman parametrization becomes simpler, due to the more local mass distribution:

$$I_9 = \tilde{K}_9 S_\varepsilon^3 e^{-\gamma_E 3\varepsilon/2}\Gamma\left(2-\frac{3\varepsilon}{2}\right)\sum_{j=0}^{N}\int_{[0,1]^7} dx\; dz\; du\; dw\; da\; ds\; dt\; z^{\frac{\varepsilon}{2}-1}(1-z)^{\frac{\varepsilon}{2}} w^{1-\varepsilon}(1-w)^{2-\varepsilon}\theta(1-s-t)(1-s-t)s^{-\frac{\varepsilon}{2}}t^{\varepsilon-2}\times$$
$$\times((1-w)u+w((1-s-t)a+sx+tu))^j(1-w)^{N-j}(u-a(1-s-t)-sx-tu)^{N-j} \;.$$

Having introduced binomial sums, the integrals represent rational, gamma and beta functions:

$$\begin{aligned} I_9 \;=\; & \tilde{K}_9 S_\varepsilon^3 e^{-\gamma_E 3\varepsilon/2}\Gamma(\varepsilon-1)\Gamma\left(2-\frac{3}{2}\varepsilon\right)B\left(\frac{\varepsilon}{2},1+\frac{\varepsilon}{2}\right)\sum_{i=0}^{N}\sum_{j=0}^{N-i}\binom{N-i}{j}(-1)^j\sum_{l=0}^{i+j}\binom{i+j}{l}(-1)^l \\ & \times\Gamma\left[\begin{matrix}3+j+i-\frac{\varepsilon}{2},3+i-\varepsilon,2+j-\varepsilon\\ 5+i+j-2\varepsilon,2+i+j+\frac{\varepsilon}{2}\end{matrix}\right]\frac{1}{(l+1)(l+2)(N+1-l)} \\ & \times\left[B\left(-\frac{\varepsilon}{2},1\right)-B\left(l+2-\frac{\varepsilon}{2},1\right)-B\left(-\frac{\varepsilon}{2},l+3\right)\right] \end{aligned}$$

Example 2: Results

After expanding in ε the sums are again performed using Sigma:

$$
\begin{aligned}
I_9 = {} & \frac{\tilde{K}_9 S_\varepsilon^3}{(2+N)(4+N)(5+N)}\Bigg\{\Bigg[S_1^2+3S_2+\Bigg(\frac{2(-1)^N(N^2+5N+7)}{(N+2)(N+3)^2} \\
& +\frac{2(2N^3+13N^2+27N+20)}{(N+1)(N+3)^2}\Bigg)S_1+\frac{2(-1)^N(2N^3+13N^2+29N+21)}{(N+1)(N+2)^2(N+3)^2} \\
& -\frac{2(2N^6+18N^5+57N^4+60N^3-53N^2-163N-99)}{(N+1)^2(N+2)^2(N+3)^2}\Bigg]\frac{1}{\varepsilon^2} \\
& +\frac{1}{N+3}\Bigg[\frac{(N+3)}{2}S_1^3+\Bigg((-1)^N\frac{(N^2+5N+7)}{2(N+2)(N+3)} \\
& +\frac{2N^6+43N^5+360N^4+1529N^3+3524N^2+4218N+2048}{2(N+1)(N+2)(N+3)(N+4)(N+5)}\Bigg)S_1^2 \\
& +\Bigg(\frac{P_{12}(N)}{(N+1)^2(N+2)(N+3)^2(N+4)(N+5)}+\frac{(-1)^N P_{13}}{(N+1)^2(N+2)^2(N+3)^2(N+4)(N+5)} \\
& +4S_{-2}(N)\Bigg)S_1+(-1)^N\frac{P_{11}}{(N+1)^3(N+2)^3(N+3)^2(N+4)(N+5)} \\
& +\frac{P_{10}}{(N+1)^2(N+2)^3(N+3)^2(N+4)(N+5)}+\frac{4(2N+3)}{(N+1)(N+2)}S_{-2} \\
& +\Bigg((-1)^N\frac{(N^2+5N+7)}{2(N+2)(N+3)}+\frac{-10N^6-133N^5-612N^4-915N^3+1052N^2+4246N+3104}{2(N+1)(N+2)(N+3)(N+4)(N+5)} \\
& +\frac{7}{2}(N+3)S_1\Bigg)S_2+2(N+5)S_3-4(N+3)S_{2,1}\Bigg]\frac{1}{\varepsilon}\Bigg\} \\
& +\frac{1}{(N+2)(N+3)(N+4)(N+5)}\Bigg\{\frac{7(N+3)}{48}S_1^4+\Bigg[\frac{(-1)^N(N^2+5N+7)}{12(N+2)(N+3)} \\
& +\frac{2N^6+59N^5+588N^4+2805N^3+7040N^2+8974N+4544}{12(N+1)(N+2)(N+3)(N+4)(N+5)}\Bigg]S_1^3 \\
& +\Bigg[\frac{(-1)^N P_{15}(N)}{4(N+1)^2(N+2)^2(N+3)^2(N+4)(N+5)} \\
& +\frac{P_{14}(N)}{4(N+1)^2(N+2)^2(N+3)^2(N+4)^2(N+5)^2}+7S_{-2}\Bigg]S_1^2 \\
& +\Bigg[\frac{(-1)^N P_{16}(N)}{2(N+1)^3(N+2)^3(N+3)^3(N+4)^2(N+5)^2} \\
& +\frac{P_{17}(N)}{2(N+1)^3(N+2)^2(N+3)^3(N+4)^2(N+5)^2}+5S_{-3} \\
& -\frac{2(5N^5+49N^4+104N^3-285N^2-1213N-1036)}{(N+1)(N+2)(N+3)(N+4)(N+5)}S_{-2}\Bigg]S_1+\frac{(55N+141)}{16}S_2^2 \\
& +\frac{(-1)^N P_{18}(N)}{2(N+1)^4(N+2)^4(N+3)^3(N+4)^2(N+5)^2} \\
& +\frac{P_{19}(N)}{2(N+1)^4(N+2)^4(N+3)^3(N+4)^2(N+5)^2} \\
& +\frac{5(2N+3)}{(N+1)(N+2)}S_{-3}-\frac{4(5N^6+63N^5+275N^4+425N^3-160N^2-1004N-684)}{(N+1)^2(N+2)^2(N+3)(N+4)(N+5)}S_{-2} \\
& +\Bigg(\frac{3}{8}(9N+31)S_1^2+\Bigg(\frac{13(-1)^N(N^2+5N+7)}{4(N+2)(N+3)} \\
& +\frac{-10N^6-65N^5+420N^4+5213N^3+18860N^2+29514N+16976}{4(N+1)(N+2)(N+3)(N+4)(N+5)}\Bigg)S_1 \\
& +\frac{(-1)^N P_{20}(N)}{4(N+1)^2(N+2)^2(N+3)^2(N+4)(N+5)} \\
& +\frac{P_{21}(N)}{4(N+1)^2(N+2)^2(N+3)^2(N+4)^2(N+5)^2}+S_{-2}\Bigg)S_2 \\
& +\zeta_2\Bigg[\frac{3}{8}(N+3)S_1^2+\Bigg(\frac{3(-1)^N(N^2+5N+7)}{4(N+2)(N+3)}+\frac{3(2N^3+13N^2+27N+20)}{4(N+1)(N+3)}\Bigg)S_1 \\
& +\frac{3(-1)^N(2N^3+13N^2+29N+21)}{4(N+1)(N+2)^2(N+3)}-\frac{3(2N^6+18N^5+57N^4+60N^3-53N^2-163N-99)}{4(N+1)^2(N+2)^2(N+3)} \\
& +\frac{9}{8}(N+3)S_2\Bigg]+\Bigg(\frac{(-1)^N(N^2+5N+7)}{6(N+2)(N+3)}+\frac{-34N^5-383N^4-1379N^3-1280N^2+1830N+2632}{6(N+1)(N+2)(N+3)(N+4)} \\
& +\frac{(13N+105)}{6}S_1\Bigg)S_3+\frac{(53-N)}{8}S_4+\Bigg(-\frac{6(2N+3)}{(N+1)(N+2)}-6S_1\Bigg)S_{-2,1} \\
& +\Bigg(\frac{12N^5+140N^4+546N^3+725N^2-93N-532}{(N+1)(N+2)(N+4)(N+5)} \\
& -(4N+15)S_1\Bigg)S_{2,1}+(N-11)S_{3,1}+(N+9)S_{2,1,1}\Bigg\}+O(\varepsilon)\,.
\end{aligned}
$$

5. Automation

So far the scalar parts of three loop ladder graphs have been calculated, which contribute to the massive OME A_{Qg}.

- The full QCD contribution requires: Dirac and color structure have to be included.
- This leads to many terms with mild variation.

$\rightarrow$ An algorithmic approach based on computer algebraic methods is developed.

The graphs are generated in an automated way, also providing comparisons for a series of finite moments [Bierenbaum, Blümlein, Klein 2009].
We develop a FORM [Vermaseren 2000] program to transform a given graph into nested sums over hypergeometric terms equipped with harmonic sums. The summation of these nested sums is carried out by Sigma [C. Schneider 2007] using modern summation technologies based on $\Pi\Sigma$-field construction.

Graph generation → Feynman Parametrization

- The graphs are generated by QGRAF [Nogueira 1993] + local operator insertions [Bierenbaum, Blümlein, Klein 2009] .
- An implementation of the Feynman rules constructs the momentum integral.
- Color traces are performed using the color package by [van Ritbergen, Schellekens, Vermaseren 1999].
- Feynman parametrization is introduced for each loop momentum

$$\prod_i \frac{1}{A_i^{a_i}} = \frac{\Gamma(\sum_i a_i)}{\prod_i \Gamma(a_i)} \int_0^1 \prod_i dx_i \, x_i^{a_i - 1} \frac{\delta(1 - \sum_i x_i)}{(\sum_i x_i A_i)^{\sum_i a_i}} \; , \quad (i = 1...n) \; .$$

- Momentum integrals are performed with standard methods.

Integral Representation

- Map the integration region to a hypercube: Integrating the δ-functions yields Heaviside functions $\theta(1-\sum_i x_i)$ which are eliminated via successive application of the relation

$$\int_0^1 dx\, \theta(1-R-x) f(x) = \int_0^1 dx\, (1-R)\, \theta(1-R)\, f((1-R)x)\ .$$

- This map may be applied in such a way that the operator insertion is mapped to a polynomial; e.g.

$$\theta(1-x_1-x_2)\theta(1-y_1-y_2)\left(x_1 + x_2 z_1 + \frac{x_2 y_1 z_2}{1-y_2} + \frac{x_1 x_2 z_3}{1-x_2}\right)^N$$

$$\longrightarrow (x_1(1-x_2) + x_2 z_1 + x_2 y_1 z_2 + x_1 x_2 z_3)^N\ .$$

- Perform integrals over certain rational structures.

- Apply, if possible, hypergeometric representations.

- Functions are introduced in a regularized form; i.e.
  - split manifestly divergent parts off, which cancel in the full graph; e.g. $(0 \le a \le b)$

$$\int_0^1 dx \sum_{s=a}^{b} x^{\varepsilon}(1-x)^{s-c} = \sum_{s=a}^{b} \theta(s-c) B(1+\varepsilon, s-c+1) + \int_0^1 \tilde{dx} \sum_{s=a}^{b} \theta(c-1-s) x^{\varepsilon}(1-x)^{s-c} \,,$$

with the θ-function defined as $\theta(a) = \begin{cases} 1 & \text{if } a \ge 0 \\ 0 & \text{if } a < 0 \end{cases}$.

  - θ-functions are then simplified using the obvious commutation relation

$$\sum_{s=a}^{b} \theta(s-c) = \theta(a-c) \sum_{s=a}^{b} + \theta(c-a-1)\theta(b-c) \sum_{s=c}^{b} .$$

- Finally binomial sums are introduced.

ε-Expansion

- Represent beta functions in terms of gamma functions.
- Again split singlular expansions ($a \geq 0$)

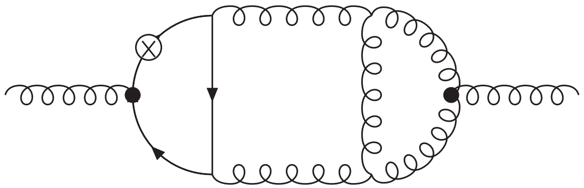
$$\Gamma(-a+\varepsilon) = \frac{(-1)^a}{\varepsilon} \frac{\Gamma(1+\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(a+1-\varepsilon)}$$

- from regular expansion

$$\Gamma(a+\varepsilon) = \Gamma(a) e^{-\gamma_E \varepsilon} \left\{ 1 + \varepsilon S_1(a-1) + \frac{\varepsilon^2}{2} \left(S_1^2(a-1) - S_2(a-1) + \zeta_2 \right) \right\} + \ldots$$

using θ functions and the commutiation relation with sums as shown above.

Example



(scalar graph, with $l = 1$ for $1/(k^2 - m^2)^l$)

$$= K_{6a} \int \hat{d}k\, \hat{d}q\, \hat{d}l \frac{(\Delta.k)^{N-1}}{((k-p)^2 - m^2)((k-l)^2 - m^2)(k^2 - m^2) l^2 (l-p)^2 (q-l)^2 (q-p)^2 q^2}$$

$$= \frac{i\tilde{K}_{6a} S_\varepsilon^3}{(N+1)(N+3)(N+4)} \Bigg\{ 4\left[S_1 - \frac{N^2+N-1}{(N+1)(N+2)}\right]\frac{1}{\varepsilon^2}$$

$$+ \left[\frac{5}{2}S_1^2 - \frac{1}{2}S_2 + \frac{-5N^4 - 18N^3 + 62N^2 + 289N + 244}{(N+1)(N+2)(N+3)(N+4)}S_1 + \frac{P_3(N)}{(N+1)^2(N+2)^2(N+3)(N+4)}\right]\frac{1}{\varepsilon}$$

$$+ \Bigg[\frac{11}{12}S_1^3 + \frac{(-8N^4 + 3N^3 + 335N^2 + 994N + 736)}{4(N+1)(N+2)(N+3)(N+4)}S_1^2 + \left(\frac{P_4(N)}{2(N+1)^2(N+2)^2(N+3)^2(N+4)^2} + \frac{11}{4}S_2\right)S_1$$

$$+ \frac{P_5(N)}{(N+1)^3(N+2)^3(N+3)^2(N+4)^2} + \frac{3}{2}\left(S_1 - \frac{(N^2+N-1)}{(N+1)(N+2)}\right)\zeta_2 + \frac{-2N^4 + 9N^3 + 185N^2 + 580N + 472}{4(N+1)(N+2)(N+3)(N+4)}S_2$$

$$- \frac{8}{3}S_3 + 6S_{2,1}\Bigg]$$

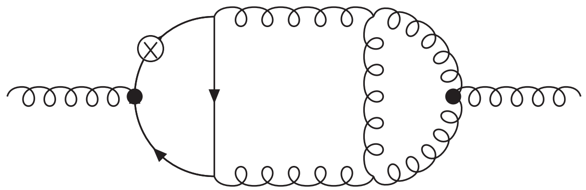
$$P_3(N) = -3N^6 - 65N^5 - 415N^4 - 1109N^3 - 1276N^2 - 468N + 64$$

$$P_4(N) = -12N^8 - 311N^7 - 2943N^6 - 13584N^5 - 32101N^4 - 32407N^3 + 7542N^2 + 40744N + 22784$$

$$P_5(N) = -24N^9 - 604N^8 - 6089N^7 - 32820N^6 - 104549N^5 - 202546N^4 - 232976N^3 - 143560N^2 - 32816N + 3328$$

obtained by Sigma

Example



(full expression including Dirac and color structure)

Here sums of the following type occur:

$$\sum_{s_1=1}^{N}\binom{N}{s_1}\sum_{s_2=0}^{s_1}\binom{s_1}{s_2}\frac{(-1)^{s_2}}{2(s_2+1)}\Gamma\left[\begin{matrix}N-s_1+2, N-s_1+3, s_1, s_2+2\\ N+3, N-s_1+s_2+4\end{matrix}\right]S_1(s_1+1)S_2(s_1+1)$$

$$= \frac{1}{2}\Bigg[\frac{S_1^2(N)}{2(N+1)(N+2)(N+3)}-\frac{2N^2+5N+4}{(N+1)^3(N+2)(N+3)}S_1(N)+\frac{N\left(3N^2+9N+7\right)}{(N+1)^3(N+2)(N+3)}$$

$$+\left(\frac{S_1^2(N)}{2(N+2)(N+3)}-\frac{N}{(N+1)(N+2)(N+3)}S_1(N)-\frac{3}{2(N+1)(N+2)(N+3)}\right)S_2(N)$$

$$+\frac{S_3(N)}{(N+1)(N+2)}-\frac{S_4(N)}{2(N+2)(N+3)}-\frac{S_{2,1}(N)}{(N+1)(N+2)}+\frac{S_{3,1}(N)}{(N+2)(N+3)}-\frac{S_{2,1,1}(N)}{(N+2)(N+3)}\Bigg]$$

obtained by Sigma

Example

$$\sum_{s_1=0}^{N-2}\binom{N-2}{s_1}\sum_{s_2=0}^{N-s_1-2}\binom{N-s_1-2}{s_2}\sum_{s_3=0}^{s_2}\binom{s_2}{s_3}\frac{(-1)^{s_3}(N-1)}{2(s_1+2)(s_3+1)(s_1+s_2+3)(s_1+s_2+2)}$$

$$\times\Gamma\left[\begin{matrix} s_1+2, N-s_1-s_2, N-s_1-s_2+1, s_2+2, s_3+1 \\ N+3, N-s_1-s_2+s_3+2 \end{matrix}\right] S_1(s_1+s_2+1)$$

$$= \frac{1}{2}\Bigg[-\frac{N^2+3N+5}{3N(N+1)(N+2)^2(N+3)}S_1^3(N)+\frac{P_1(N)}{2N(N+1)^3(N+2)^3(N+3)^2}S_1^2(N)+\frac{P_2(N)}{N(N+1)^3(N+2)^3(N+3)^2}S_1(N)$$

$$+\frac{3N^4+29N^3+100N^2+145N+73}{(N+1)^3(N+2)^2(N+3)^2}+\left(\frac{S_1^2(N)}{N(N+1)(N+2)}-\frac{N^4+8N^3+19N^2+9N-7}{N(N+1)^2(N+2)^2(N+3)}S_1(N)\right.$$

$$\left.+\frac{P_3(N)}{2N(N+1)^3(N+2)^3(N+3)^2}\right)S_2(N)+\left(\frac{3N^4+28N^3+82N^2+86N+23}{3N(N+1)^2(N+2)^2(N+3)}-\frac{S_1(N)}{N(N+1)(N+2)}\right)S_3(N)$$

$$+\left(-\frac{N^3+7N^2+17N+13}{N(N+1)^2(N+2)^2(N+3)}-\frac{S_1(N)}{N(N+1)(N+2)}\right)S_{2,1}(N)+\frac{S_{2,1,1}(N)}{N(N+1)(N+2)}\Bigg]$$

$$P_1(N) = -N^6-12N^5-53N^4-118N^3-151N^2-106N-27$$
$$P_2(N) = 2N^7+27N^6+148N^5+420N^4+650N^3+514N^2+142N-25$$
$$P_3(N) = -2N^7-33N^6-222N^5-789N^4-1578N^3-1725N^2-898N-141$$

obtained by Sigma

6. Summary and Outlook

- In the present project the most difficult contributions to the three loop ladder graphs, with 3 to 6 massive lines contributing to the massive OME A_{Qg} are calculated.
- This is achieved by identifying the hypergeometric structure generated from the underlying topology and by introducing appropriate finite binomial expansions.
- The sums over hypergeometric terms equipped with harmonic sums are solved using modern symbolic summation methods.
- The results are given in terms of rational expressions, harmonic sums and generalizations thereof.
- First steps of an algorithmic approach of the calculation have been presented in the case of the ladder graphs.
- The incorporation of more general hypergeometric functions is possible within this approach.
- The method will be applied to all ladder topologies contributing to the 3-loop massive Wilson coefficients in the asymptotic region.