Complex Matrix Model Duality

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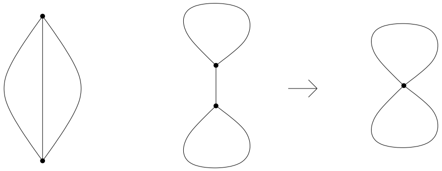
Moduli space of punctured Riemann surfaces from graphs

Theorem thanks to Mumford, Strebel, Harer and Penner in 80's:

Moduli space of genus g Reimann surfaces with s punctures decorated by boundary lengths $\mathcal{M}_{g,s} \times \mathbb{R}_+^s$ has a cell decomposition given by inequivalent ribbon graphs with s faces and lengths assigned to each edge.

Top-dim cells $\dim_{\mathbb{R}} = 6g - 6 + 3s$ come from edge-lengths of graphs with 3-valent vertices; lower-dim cells from collapsing edges to get higher-valency vertices.

E.g. for the thrice-punctured sphere $\mathcal{M}_{0,3} \times \mathbb{R}_+^3$ have graphs with 3 faces:



Including closed string operators: the Kontsevich model

Generating function of correlation functions in 2d topological gravity given by Kontsevich matrix model

$$\exp \sum_{g=0}^{\infty} \left\langle \exp \sum_k t_k \mathcal{O}_k \right\rangle_g = \rho(\Lambda)^{-1} \int [dM]^H_{n\times n} \; e^{-\frac{1}{2}\operatorname{tr}(\Lambda M^2)+\frac{i}{6}\operatorname{tr}(M^3)}$$

The coupling coefficients are encoded in the constant matrix Λ

$$t_k = -\sum_{i=1}^{n} \frac{1}{k\lambda_i^k} = -\frac{1}{k}\operatorname{tr}(\Lambda^{-k})$$

The colour index for each face gets associated to eigenvalues λ_i and hence to the couplings t_k. The propagator can be transformed into an integral over the corresponding edge length p using the Schwinger trick

$$\left\langle M^i_j \; M^k_l \right\rangle \;=\; \delta^i_l \delta^k_j \frac{2}{\lambda_i + \lambda_j} \;=\; 2\delta^i_l \delta^k_j \int_0^\infty dp \; e^{-p\,(\lambda_i+\lambda_j)}$$

The $\mathbb{C}$ Z matrix model

The Z model for an $N \times N$ $\mathbb{C}$ matrix (N large but finite) is

$$\mathcal{Z}(\{t\},\{\bar{t}\}) = \int [dZ]^{\mathbb{C}}_{N\times N} \ e^{-\operatorname{tr}(ZZ^\dagger)+\sum_{k=1}^{\infty} t_k \operatorname{tr}(Z^k)+\sum_{k=1}^{\infty} \bar{t}_k \operatorname{tr}(Z^{\dagger k})}$$

It generates the full genus expansions of

- Extremal correlation functions of certain half-BPS local operators for free $4d, \mathcal{N}=4$ super Yang-Mills with gauge group $U(N)$.
- Scattering of integer-momentum tachyons in $c=1$ string at self-dual radius, cosmological constant $\mu = iN$. The map to tachyons is $\mathcal{T}_k \to \operatorname{tr}(Z^k)$ and $\mathcal{T}_{-k} \to \operatorname{tr}(Z^{\dagger k})$.

The individual correlation functions are

$$\left\langle \operatorname{tr}(Z^{k_1}) \cdots \operatorname{tr}(Z^{k_p}) \ \ \operatorname{tr}(Z^{\dagger \bar{k}_1}) \cdots \operatorname{tr}(Z^{\dagger \bar{k}_q}) \right\rangle$$

How do we rewrite this model so that closed string insertions associate with faces rather than vertices?

The dual $\mathbb{C}$ F matrix model

The dual F model is the same function of the couplings $\{t\}, \{\bar{t}\}$

$$\mathcal{Z}(\{t\},\{\bar{t}\}) = \int [dF]^{\mathbb{C}}_{n\times n} \; e^{-\operatorname{tr}(AFBF^\dagger)+N\sum_{k=1}^{\infty}\frac{1}{k}\operatorname{tr}\left[\left(FF^\dagger\right)^k\right]}$$

The couplings $\{t\}$ and $\{\bar{t}\}$ are encoded in constant matrices A, B

$$t_k = \sum_{i=1}^{n}\frac{1}{ka_i^k} = \frac{1}{k}\operatorname{tr} A^{-k} \qquad \bar{t}_k = \sum_{j=1}^{n}\frac{1}{kb_j^k} = \frac{1}{k}\operatorname{tr} B^{-k}$$

The colour index for each face of the F model Feynman diagrams comes with either a_i's or b_j's, so the couplings $\{t\}$ and $\{\bar{t}\}$ are associated to faces (dual to vertices of Z model)

$$\left\langle F^i_j \; F^{\dagger}{}^k_l \right\rangle = \frac{\delta^i_l \delta^k_j}{(a_i b_j - N)}$$

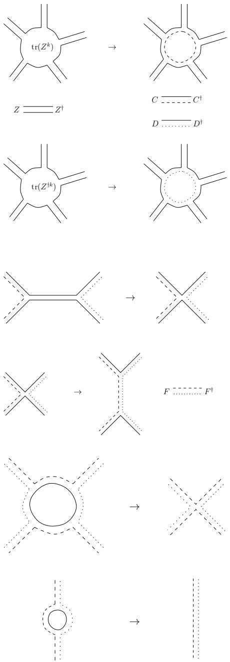
$$\int [dZ]\; e^{-\,\mathrm{tr}(ZZ^\dagger)}\, \frac{1}{\det(A\otimes\mathbb{I}_N-\mathbb{I}_n\otimes Z)}\, \frac{1}{\det(B\otimes\mathbb{I}_N-\mathbb{I}_n\otimes Z^\dagger)}$$

$$= \int [dZ, C, D]\; e^{-\,\mathrm{tr}\left[ZZ^\dagger + C^\dagger A C - C^\dagger Z C + D^\dagger B D - D^\dagger Z^\dagger D\right]}$$

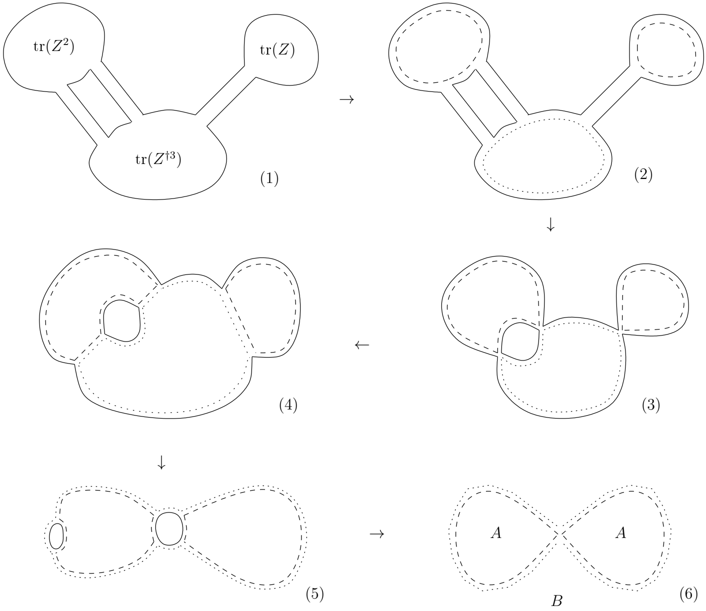
$$= \int [dC, D]\; e^{-\,\mathrm{tr}\left[C^\dagger A C + D^\dagger B D - CC^\dagger DD^\dagger\right]}$$

$$= \int [dF, C, D]\; e^{-\,\mathrm{tr}\left[FF^\dagger - D^\dagger F^\dagger C - C^\dagger F D + C^\dagger A C + D^\dagger B D\right]}$$

$$= \int [dF]\; e^{-\,\mathrm{tr}(AFBF^\dagger) + N\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{tr}\left[\left(FF^\dagger\right)^k\right]}$$

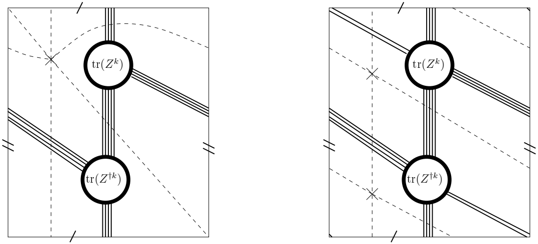


Example in action



Calculation: two-point function on the torus

Z model graph is solid line; dual F model graph is dashed line



Each F model graph captures infinitely many bunched Z graphs

$$\left\langle \mathrm{tr}(FF^\dagger FF^\dagger FF^\dagger) \right\rangle_{\mathrm{torus}} + \left\langle \mathrm{tr}(FF^\dagger FF^\dagger) \;\; \mathrm{tr}(FF^\dagger FF^\dagger) \right\rangle_{\mathrm{torus}}$$

$$= \sum_{k=3}^{\infty} t_k \bar{t}_k k \left[\binom{k}{3} + \binom{k}{4} \right] N^{k-2}$$

$$= \sum_{k=3}^{\infty} t_k \bar{t}_k \left\langle \mathrm{tr}(Z^k) \;\; \mathrm{tr}(Z^{\dagger k}) \right\rangle_{\mathrm{torus}}$$

Discrete Schwinger parameterisation of F matrix model

Following analysis of Chekhov-Makeenko model (the Hermitian version of the F model), rewrite $a_i = \sqrt{N} e^{\varepsilon l_i}$ and $b_j = \sqrt{N} e^{\varepsilon m_j}$ with discretisation parameter ε. The propagator becomes

$$\frac{1}{a_i b_j - N} = \frac{1}{N e^{\varepsilon l_i} e^{\varepsilon m_j} - N} = \frac{1}{N} \sum_{p=1}^{\infty} e^{-p\varepsilon(l_i + m_j)}$$

The sum is a discrete Schwinger parameterisation of the edge length for the propagator. Each summand comes from an edge of integer length $p\varepsilon$.

Edge lengths correspond to coördinates on (subspaces of) the moduli space of Riemann surfaces, so F model correlation functions localise on discrete points in the moduli space. For each dual Z correlation function there are furthermore only a finite number of points.

[This is no surprise since the correlation functions count very particular holomorphic Belyi maps from algebraic Riemann surfaces to $\mathbb{CP}^1$, whose Strebel differentials have integer-length critical graphs.]

Conclusions

- Same complex matrix model (Z model) generates tachyon scattering for $c = 1, R = 1$ string and computes correlation functions of half-BPS operators in free $d = 4, \mathcal{N} = 4$ SYM. Closed string insertions appear as vertices of the Feynman diagrams.
- There is a dual complex matrix model (F model) where now closed string insertions are associated to faces of the dual diagrams.
- Using a discrete Schwinger parameterisation of the F model propagators, the correlation functions localise on discrete points in (subspaces of) the moduli space of Riemann surfaces.

Questions:

- What is the geometric relation between the $c = 1, R = 1$ string and this sector of (small radius) AdS_5/CFT_4?
- How does the dual worldsheet theory localise on the moduli space?
- Can correlation functions of other operators of (free) $\mathcal{N} = 4$ SYM be treated this way to gain a microscopic understanding of AdS/CFT?