Affine $SL(N)$ conformal blocks from $\mathcal{N}=2$ $SU(N)$ gauge theories

Filippo Passerini

Institut für Physik, Humboldt-Universität zu Berlin

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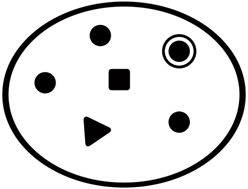
C. Kozcaz, S. Pasquetti, F.P. and N. Wyllard, arXiv:1008.1412 [hep-th]

Outline

A large class of 4-dimensional $\mathcal{N} = 2$ gauge theories

Gaiotto has constructed a large class of 4-dimensional $\mathcal{N} = 2$ gauge theories that describe the low energy dynamics of a stack of N M5-branes compactified on a punctured Riemann surface $C_{(f_a),g}$ [Gaiotto 09] [Witten 97]



A $\mathcal{N} = 2$ gauge theory is associated to any punctured Riemann surface

$$C_{(f_a),g} \quad \Longleftrightarrow \quad \mathcal{T}_{(f_a),g}$$

$\mathcal{T}_{(f_a),g}$ is characterized by the same data labeling the surface

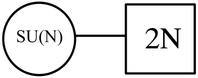
- the genus g
- the number of punctures (f_a)

More in details

- (f_a) punctures encode the flavor symmetry of the gauge theory $\mathcal{T}_{(f_a),g}$
- the different degenerations of the surface $C_{(f_a),g}$ such that it becomes a set of pairs of pants connected by thin tubes are associated to the different S-duality frame of the gauge theory $\mathcal{T}_{(f_a),g}$
- the thin tubes connecting the pair of pants are the weakly coupled gauge groups

$\mathcal{N}$=2 $SU(N)$ gauge theory with $N_f = 2N$

Let's consider the conformal $\mathcal{N}$=2 $SU(N)$ gauge theory coupled to $N_f = 2N$ hypermultiplets

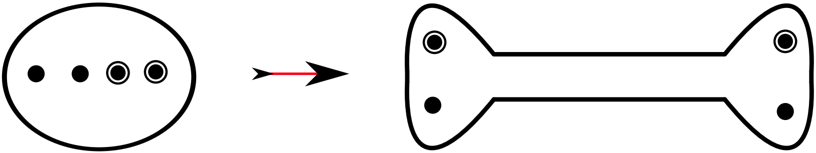


The flavor group is $U(1) \otimes SU(N) \oplus U(1) \otimes SU(N)$. There are two types of flavor group thus we need two types of punctures:

- simple puncture associated to $U(1)$ flavor group
- full puncture associated to $SU(N)$ flavor group

$C_{(2,2),0}$ Riemann surface

The weakly coupled conformal $\mathcal{N}$=2 $SU(N)$ gauge theory coupled to $N_f = 2N$ hypermultiplets is associated to a degeneration of $C_{(2,2),0}$



The tube in the center is associated to the $SU(N)$ gauge group

- One of the possible degenerations describes a strongly coupled theory that does not admit a Lagrangian description [AS 07]
- Different theories associate to different degeneration of $C_{(2,2),0}$ are related by the S-duality transformations
- Geometrical interpretation of S-duality

It is possible to use the 2D-4D connection to perform quantitative computations!! [AGT 09]

From 4D to 2D: the AGT proposal

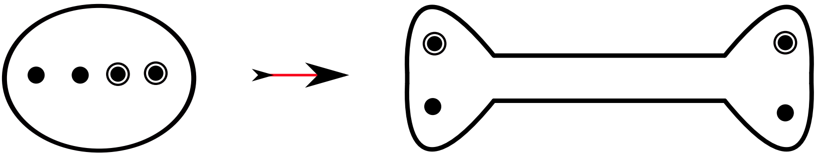
The partition function of four dimensional gauge theories $\mathcal{T}_{(f_a),g}(A_{N-1})$ defined on S^4 is equivalent to a correlator of two dimensional A_{N-1} Toda field theory defined on $C_{(f_a),g}$ [AGT 09][Wyllard 09]

$$Z_{\mathcal{T}_{f,g}} = \langle V_{m_1} \dots V_{m_f} \rangle_{A_{N-1} \textit{ Toda on } C_{(f_a),g}}$$

- there is one primary for each puncture and the momenta $m_1, \dots, m_f$ are related to the masses of the hypermultiplets

$C_{(2,2),0}$ Riemann surface

E.G. Consider the case of $C_{(2,2),0}$



Different types of punctures are associated to different types of $\mathcal{W}_N$ primaries

- simple puncture associated to semidegenerate state V_χ with $\chi = \kappa\omega_1$
- full puncture associated to non-degenerate state V_m with generic m

The correlator

$$\langle m_1 | V_{\chi_2}(1) V_{\chi_3}(z, \bar{z}) | m_4 \rangle$$

$$= \int d\alpha \langle m_1 | V_{\chi_2}(1) | \alpha \rangle \langle \alpha | V_{\chi_3}(z, \bar{z}) | m_4 \rangle \mathcal{F}^{(\sigma)}_{\alpha, m_1, \chi_2, \chi_3, m_4}(z) \bar{\mathcal{F}}^{(\sigma)}_{\alpha, m_1, \chi_2, \chi_3, m_4}(\bar{z})$$

reproduce the partition function for the $\mathcal{N}$=2 $SU(N)$ theory with $N_f = 2N$ hypermultiplets!

For these gauge theories $Z_{\mathcal{T}_{f,g}} = \int [da]\, \tilde{Z}^{(\sigma)}\, \bar{\tilde{Z}}^{(\sigma)}$ where $\tilde{Z} = Z_{\text{pert}} Z_{\text{instanton}}$
[Pestun 07]

AGT has pointed out that

$$Z^{(\sigma)}_{\text{instanton}}(\tau, a, \hat{m}, \epsilon_1, \epsilon_2) \sim \mathcal{F}^{(\sigma)}_{\alpha,m}(z)$$

$$Z_{\text{pert}} \sim \text{3-point functions}$$

Considering

- $a \sim \alpha$ VEV of the scalars are related to the internal momenta
- $\hat{m} \sim m$ masses of the hypers are related to external momenta
- $e^{2\pi i\tau} = z$ gauge couplings are related to the worldsheet coordinates

M-theory picture

The $\mathcal{N} = 2$ 4D theories can be thought as the worldvolume theory of M5-branes compactified on $C_{(f_a),g}$

	4D				C						
	0	1	2	3	4	5	6	7	8	9	10
N M5	X	X	X	X	X	X					

To insert non-local operators in the gauge theory, we add to the system other M2 or M5 in order to form supersymmetric intersections

M-theory picture

Let's now consider the following M5-M5 intersection

	4D				C						
	0	1	2	3	4	5	6	7	8	9	10
N M5	X	X	X	X	X	X					
M5	X	X			X	X	X	X			

On the N M5-branes worldvolume, the intersecting M5-brane manifests itself as

- 2-dimensional operator on the 4D space $\Rightarrow$ $\frac{1}{2}$ BPS Surface Operator!
- wrap completely the $C_{f,g}$ $\Rightarrow$ different 2-dimensional CFT! [AT 10]

As argued by Alday and Tachikawa [AT 10], this M-theory setup describes a full ramified surface operator [GW 06]

full ramified surface operators

A ramified surface operator is defined imposing that the fields in the theory posses a certain singularity on the 2D subspace where the operator is supported

On the plane where a full operator is located

- the unbroken gauge group is $U(1)^{N-1}$
- $N-1$ monopole numbers $\ell_i = \frac{1}{2\pi}\int_{z_2=0} F_i$

A full ramified instanton is characterized by N topological quantities $(k, \ell_1, \ldots, \ell_{N-1})$

Instanton function with surface operators

The instanton partition function is given by [Braverman 04][BE 04][Negut 08][FFNR 08][AT 10]

$$Z_{instanton} = \sum_{\lambda} Z_{\vec{k}}(\lambda) \prod_{i} y_i^{k_i}$$

- $k_1 = k \qquad k_{i+1} = k_i + \ell_i$
- $\lambda = (\lambda_1, \ldots, \lambda_N)$ is a vector of Young tableau
- $k_i = \sum_{j \geq 1} \lambda_j^{i-j+1}$
- $Z_{\vec{k}}(\lambda)$ depends on the field content

$\mathcal{N}=2$ $SU(N)$ gauge theory with $N_f=2N$

Let's focus on the $\mathcal{N}=2$ $SU(N)$ gauge theory with $N_f=2N$ hypers with a full ramified surface operator. We define [KPPW 10]

- $Z^{(0),i}$ the sum of all terms with $k_i\neq 0$ and $k_j=0$ for $i\neq j$
- $Z^{(1),i,j}$ the sum of all terms with $k_i\neq 0$, $k_j=1$ for and $k_r=0$ for $r\neq i,j$

Thus

$$Z_{instanton}=\sum_i Z^{(0),i}+\sum_{i,j}Z^{(1),i,j}\dots$$

where

$$Z^{(0),i}=\sum_{n=1}^{\infty}\frac{(\frac{\mu_{i+1}}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor+1)_n(\frac{\tilde{\mu}_i}{\epsilon_1}-\frac{a_i}{\epsilon_1})_n}{(\frac{a_{i+1}}{\epsilon_1}-\frac{a_i}{\epsilon_1}+\frac{\epsilon_2}{\epsilon_1}\lfloor\frac{i}{N}\rfloor+1)_n\,n!}(-y_i)^n$$

- a_i are the Coulomb branch parameters
- $\mu_i,\tilde{\mu}_i$ are the hypermultiplets masses

$\mathcal{N} = 2$ $SU(2)$ theories and affine $SL(2)$ algebra

The instanton partition function for $\mathcal{N} = 2$ $SU(2)$ gauge theories with a full ramified surface operator are equivalent to modified affine $SL(2)$ conformal blocks [AT 10]

Affine $SL(2)$ algebra is defined by

$$[J_n^0, J_m^0] = \frac{k}{2}\, n\, \delta_{n+m,0}\,, \quad [J_n^0, J_m^\pm] = \pm J_{n+m}^\pm\,, \quad [J_n^+, J_m^-] = 2J_{n+m}^0 + k\, n\, \delta_{n+m,0}$$

- $|j\rangle$ primary state, $J_0^0|j\rangle = j|j\rangle$ and $J_{1+n}^-|j\rangle = J_{1+n}^0|j\rangle = J_n^+|j\rangle = 0$
- $V_j(x,z)$ primary field, x is an isospin variable and z is the worldsheet coordinate

The generators act on the primary fields as differential operators

$$[J_n^A, V_j(x,z)] = z^n D^A V_j(x,z)$$

$$D^+ = 2j\,x - x^2\partial_x\,, \qquad D^0 = -x\partial_x + j\,, \qquad D^- = \partial_x$$

E.G. for the $\mathcal{N}=2$ $SU(2)$ gauge theory with $N_f=4$ hypermultiples it results

$$Z_{\text{instanton}} = (1-z)^{2j_2(-j_3+k/2)} \langle j_1 | \mathcal{V}_{j_2}(1,1) \mathcal{V}_{j_3}(x,z) | j_4 \rangle$$

where

$$\mathcal{V}_j = \mathcal{K} V_j \qquad \mathcal{K}(x,z) = \exp\left[-\sum_{n=1}^{\infty} \frac{1}{2n-1} \left(z^{n-1} x\, J^-_{1-n} + \frac{z^n}{x} J^+_{-n} \right) \right]$$

It can be evaluated pertubatively considering the decomposition

$$\sum_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'} \langle j_1 | \mathcal{V}_{j_2}(1,1) | \mathbf{n},\mathbf{A}; j \rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j) \langle \mathbf{n}',\mathbf{A}'; j | \mathcal{V}_{j_3}(x,z) | j_4 \rangle$$

The components $Z^{(0),i}$ of the instanton function are reproduced by terms with the following internal states [KPPW 10]

- $(J_0^-)^n |j\rangle$ that gives a x^n term
- $(J_{-1}^+)^n |j\rangle$ that gives a $\left(\frac{z}{x}\right)^n$ term

The complete dictionary is

- $y_1 = x\,, \qquad y_2 = \frac{z}{x}\,, \qquad j = -\frac{1}{2} + \frac{a_1}{\epsilon_1}\,, \quad k = -2 - \frac{\epsilon_2}{\epsilon_1}$
- $j_1 = -\frac{\epsilon_1+\epsilon_2+\mu_1-\mu_2}{2\epsilon_1}\,, \qquad j_2 = -\frac{2\epsilon_1+\epsilon_2+\mu_1+\mu_2}{2\epsilon_1}$
- $j_3 = -\frac{2\epsilon_1+\epsilon_2-\tilde{\mu}_1-\tilde{\mu}_2}{2\epsilon_1}\,, \quad j_4 = -\frac{\epsilon_1+\epsilon_2+\tilde{\mu}_1-\tilde{\mu}_2}{2\epsilon_1}$

What about the $SU(N)$ gauge theories with N>2 ? [KPPW 10]

$\mathcal{N} = 2$ $SU(N)$ theories and affine $SL(N)$ algebra

Affine $SL(N)$ algebra is generated by

$$J^i_n\,, \qquad J^{i+}_n\,, \qquad J^{i-}_n\,, \qquad J^{il}_n \quad (i \neq l)$$

- $|j\rangle$ primary state, is labeled by $j = \sum_{i=1}^{N-1} j^i \omega_i$ where ω_i are the fundamental weights of $SL(N)$

- $J^i_0|j\rangle = j^i|j\rangle$ and $J^{i+}_0|j\rangle = 0\,, \quad J^{il}_0|j\rangle = 0 \ \ (i > l)\,, \quad J^A_n|j\rangle = 0 \ \ (n > 0)$

- $V_j(x,z)$ primary field, x is a vector of isospin variables and z is the worldsheet coordinate

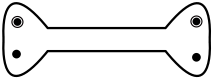
The generators act on the primary fields as differential operators

$$[J^A_n, V_j(x,z)] = z^n D^A V_j(x,z)$$

In general, x is a vector of $\frac{N(N-1)}{2}$ isospin variables and D^A are differential operators in these variables. Too many isospin variables!

The primary field V_χ with $\chi = \kappa\omega_1$ depends only on $N-1$ isospin variables and the action of the generators on these fields is expressed in terms of differential operators D^A that depends on $N-1$ isospin variables

Let's focus on the conformal $\mathcal{N}$=2 $SU(N)$ gauge theory coupled to $N_f = 2N$ hypermultiplets, i.e.



- simple puncture associated to a state V_χ
- full puncture associated to a state V_j with generic j

The instanton function is equivalent to

$$\sum_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'} \langle j_1|\mathcal{V}_{\chi_2}(1,1)|\mathbf{n},\mathbf{A};j\rangle X^{-1}_{\mathbf{n},\mathbf{A};\mathbf{n}',\mathbf{A}'}(j)\langle \mathbf{n}',\mathbf{A}';j|\mathcal{V}_{\chi_3}(x,z)|j_4\rangle$$

where

$$\mathcal{V}_{\chi_i}(x,z) = V_{\chi_i}(x,z)\mathcal{K}^{\dagger}(x,z)$$

and the dictionary is

- $y_1 = x_1\,, \qquad y_{i+1} = \frac{x_{i+1}}{x_i} \quad (1 \le i \le N-2)\,, \qquad y_N = \frac{z}{x_{N-1}}$
- $j^i = -\frac{1}{2} + \frac{a_i - a_{i+1}}{2\epsilon_1}\,, \qquad k = -N - \frac{\epsilon_2}{\epsilon_1}$
- $\frac{\tilde{\mu}_i}{2\epsilon_1} = -\frac{\kappa_3}{N} + \langle h_i, j_4 + \frac{\rho}{2}\rangle\,, \qquad \frac{\mu_i}{2\epsilon_1} = \frac{\kappa_2}{N} + \langle h_i, j_1 + \frac{\rho}{2}\rangle$

Conclusions

- We extended the AT proposal to the case of conformal $SU(N)$ theories with $N > 2$
- We discussed also the AT proposal for non-conformal theories
- Can we reproduce the full partition function considering the correlator of some CFT with affine algebra?
- What is the physical meaning of the $\mathcal{K}$ operator?