N-Soliton solutions in Field and String Theory

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Motivation

• Classical string solutions play an important role in understanding various aspects of the AdS/CFT.

• One can use integrability methods to check and maybe even prove the correspondence.

• One particular class of classical string solutions are the solitonic ones (at the classical level we can map those solutions to solitons in sin(h)-Gordon)

• Some examples of solitonic solutions

-GKP string
-Kruczenski (application to scattering amplitudes at strong coupling)
-Berkovits-Maldacena
-Giant magnons in various spaces

We will focus on the last example

Outline

- **Describe the objects we are interested in (Giant Magnons)**
- **Describe a method to find them**
  (Dressing method (1978))
- **Find the N-magnon solution**
- **Talk about some of the properties of the N-soliton solution**
- **Conclude**

Giant magnon limit [Hofman, Maldacena (2006)]

According to the AdS/CFT dictionary states with $E-J=0$ correspond to a long chain of Z fields

$$E-J=0 \quad \Leftrightarrow \quad \mathrm{tr}(\mathrm{Z}^{\mathrm{J}}).$$

One can also consider states with finite $E-J$

$$E-J=\text{finite} \quad \Leftrightarrow \quad \mathcal{O}_p=\sum_l e^{ipl}(\ldots ZZZ\underset{\substack{\uparrow \\ l}}{W}ZZZ\ldots),$$

Using supersymmetry, Beisert has shown that

$$E-J=\sqrt{1+\frac{\lambda}{\pi^2}\sin^2\frac{p}{2}}$$

in the large coupling limit

$$E-J=\frac{\sqrt{\lambda}}{\pi}\left|\sin\frac{p}{2}\right|.$$

The giant magnon limit is defined

$$E,J\to\infty, \quad \lambda=g_{YM}^2N=\text{fixed}, \quad p=\text{fixed}, \quad E-J=\text{finite}.$$

Pohlmeyer map of giant magnons [1976]

At the classical level there is a correspondence between

$$\text{sine-Gordon solitons} \leftrightarrow \text{giant magnons}$$
$$\cos\alpha = \partial X \cdot \bar{\partial} X$$

We can compare energy, time delay, and phase shift for the two theories

sine-Gordon	giant magnon
$E_{\mathrm{sG}} = \gamma$	$E_{\mathrm{magnon}} = \frac{\sqrt{\lambda}}{\pi}\frac{1}{\gamma}$
$\Delta T_{CM} = \frac{2}{\gamma v}\log v$	$\Delta T_{12} = \frac{2}{\gamma_1 v_1}\log v_{cm}$

$$v = \cos\frac{p}{2}, \quad \gamma^{-2} = 1 - v^2, \quad \text{phase shift} = \int dE_1 \Delta T_{12}$$

The dressing method [Zakharov, Mikhailov, Shabat (1978)]

• In 1967 the inverse scattering method (ISM) was discovered by Gardner, Green, Kruskal, Miura. The task of enumerating nonlinear differential equations integrable by this method became fundamental.

• In 1978 an algorithm for constructing exact solutions of new classes of equations integrable by ISM was given. This is the so called dressing method.

• We would like to study scattering and bound states of giant magnons. A candidate method to study these states is the dressing method. In one line we can say that the dressing method generates **new soliton solutions** from old ones.

• The last years the dressing method has been successfully used in the context of giant magnons by many authors [Hollowood, Jevicki, C.K., Miramontes, Papathanasiou, Spradlin, Suzuki, Volovich, ...].

We now proceed to explain how the dressing method works.

- We start with the lagrangian of the principal chiral model

$$\mathcal{L} = \mathrm{Tr}[(\partial_\mu g g^{-1})^2]$$

where g is an element of a Lie group. For example for strings in S^3 we can take

$$g = \begin{pmatrix} X_1 + iX_2 & X_3 + iX_4 \\ X_3 - iX_4 & X_1 - iX_2 \end{pmatrix} \in SU(2),$$

where X_i are embedding coordinates of S^3.

- The eom of the principal chiral model for the matrix field $g(z,\bar{z})$ are

$$\bar{\partial}(\partial g g^{-1}) + \partial(\bar{\partial} g g^{-1}) = 0.$$

Then we are looking for a new solution of the form

$$\underbrace{g'(z,\bar{z})}_{\text{new solution}} = \underbrace{\chi(z,\bar{z})}_{\text{dressing factor}} \times \underbrace{g(z,\bar{z})}_{\text{known solution}}$$

Finding the dressing factor χ

The eom for g

$$\bar{\partial}(\partial g g^{-1}) + \partial(\bar{\partial} g g^{-1}) = 0$$

is the compatibility condition $(\partial(\bar{\partial}\Psi) = \bar{\partial}(\partial\Psi))$ of the first order linear system

$$\partial\Psi(\lambda) = \frac{\partial g g^{-1}\Psi(\lambda)}{1-\lambda}, \quad \bar{\partial}\Psi(\lambda) = \frac{\bar{\partial} g g^{-1}\Psi(\lambda)}{1+\lambda}$$

for $\lambda = 0$. The complex parameter λ is called the spectral parameter. Suppose we know a solution to the above system, $\Psi(0) = g$. We will now find a new one. We make the ansatz

$$\Psi'(\lambda) = \chi(\lambda)\Psi(\lambda), \quad \chi(\lambda) = 1 + \frac{\lambda_1 - \bar{\lambda}_1}{\lambda - \lambda_1} P,$$

where λ_1 is an arbitrary complex constant. It turns out that P is a projector onto the subspace spanned by $\Psi(\bar{\lambda}_1)e_1$ for arbitrary constant vector e_1. Then the new solution that is labeled by (λ_1, e_1) is

$$g' = \Psi'(0).$$

The N-magnon solution in $R \times S^3$

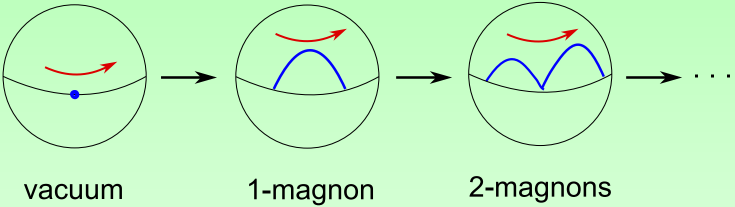

Figure: In order to find the N-magnon solution we start by applying the dressing method to the the vacuum, a point particle that moves with the speed of light around an equator of the S^3. The endpoints of the dressed solutions also move around the equator with the speed of light. Applying the dressing methods N-times we can get the N-magnon solution that corresponds to an N-magnon excitation in the gauge theory side. The angle seperation of the endpoints corresponds to the momentum of the magnon in the gauge theory side. The solution is dual to the N-soliton solution in the complex sine-Gordon model.

N-magnon solution

• The dressing method gives us (complicated) recursion relations that relate N magnons to $N-1$ magnons that we can write explicitly as

$$\Psi_N(\lambda) = \left(1 + \frac{\lambda_N - \bar{\lambda}_N}{\lambda - \lambda_N} P\right) \Psi_{N-1}(\lambda), \quad P = \frac{\Psi_{N-1}(\bar{\lambda}_N) e_N e_N^\dagger \Psi_{N-1}^{-1}(\lambda_N)}{e_N^\dagger \Psi_{N-1}^{-1}(\lambda_N) \Psi_{N-1}(\bar{\lambda}_N) e_N}$$

• The proof is based on properties of determinants (they take care of redundancy and how to go from N to $N-1$ magnons)

$$\begin{vmatrix} a_1 + \lambda b_1 & b_1 & \cdots & c_1 \\ a_2 + \lambda b_2 & b_2 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n + \lambda b_n & b_n & \cdots & c_n \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & \cdots & c_1 \\ a_2 & b_2 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & b_n & \cdots & c_n \end{vmatrix}$$

$$\begin{vmatrix} 1 & x_1 & \cdots & x_n \\ 0 & a_1 & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_n & \cdots & c_n \end{vmatrix} = \begin{vmatrix} a_1 & \cdots & c_1 \\ \vdots & \ddots & \vdots \\ a_n & \cdots & c_n \end{vmatrix}.$$

The N magnon solution for CP^n

As an example we present the N magnon solution for CP^n

$$Z_N = \left(\det(\alpha_{ij}) + \sum_{i,j=1}^{N}(-1)^{i+j} M_{ij} h_j h_i^{\dagger}\right) Z_0,$$

where

- Z_i the worldsheet coordinates for the i'th magnon,
- M_{ij} is the minor formed by removing the i'th row and j'th column of the matrix with elements α_{ij}, $i, j = 1, \ldots, N$,
- $h_i = \theta \Psi_0(\bar{\lambda}_i) e_i, \ \theta = \mathrm{diag}(-1, 1, \ldots, 1)$
- $\alpha_{ij} = -\frac{\lambda_i \beta_{ij}}{\lambda_i - \bar{\lambda}_j} - \frac{\gamma_{ij}}{\lambda_i \bar{\lambda}_j - 1}, \ \beta_{ij} = h_i^{\dagger} h_j, \ \gamma_{ij} = h_i^{\dagger} \theta \Psi(0) h_j$

where λ_j is the spectral parameter and e_i the polarization vector.

General comments on our N-magnon solution

$$Z_N = \left(\det(\alpha_{ij}) + \sum_{i,j=1}^{N} (-1)^{i+j} M_{ij} h_j h_i^\dagger\right) Z_0,$$

- We have not specified what Z_0 is. In fact it can be any solution of the string model. It can be taken for example to be the 1- or 2-magnon solution or even a non-solitonic one.
- It is a general observation that N-soliton solutions of various integrable systems can be written as $N \times N$ determinants. Therefore our formula is not of a surprise. The difficult part is to find the appropriate variables in which we can express the solution in a nice form.
- We have not found all possible solutions of the system. It may also possess not solitonic ones.

Asymptotic behavior

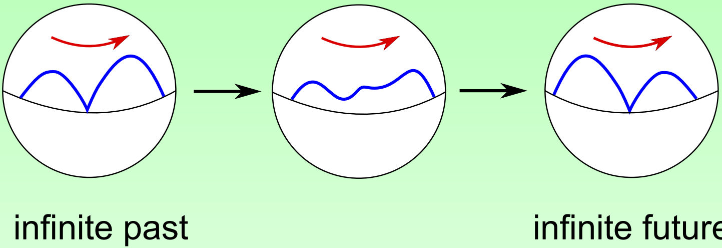

Figure: In the $x \to \pm\infty$ the magnon touches the equator with $p = \sum p_i$. In the $t \to \pm\infty$ limit we can prove that asymptotically the N-magnons split into N-single magnons, with the effect of the interaction being encoded only in a relative time delay (the shape of the magnons remain the same after scattering). The N-magnons exhibit the property of factorized scattering as expected by the integrability of the σ-model. The time delay agrees with the $\sin$-Gordon time delay. The phase shift agrees with the large λ limit of the phase in gauge theory [Arutyunov, Frolov, Staudacher].

Concluding words

- We have applied the dressing method to the spaces $SU(n),\ SO(n)/SO(n-1),\ CP^n$. The N-soliton solutions were found.
- Other spaces where the method can be applied are $Sp(n)/U(n),\ SO(n),\ SO(2n)/U(n)$ and many more.
- The knowledge of the N-soliton result may be interesting in constructing an effective Hamiltonian description of giant magnons.
- It would be interested to see what the dressing method can say in AdS.