Mathieu moonshine and elliptic genus of K3

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References

M.R.Gaberdiel, S.Hohenegger, R.Volpato, 1006.0221; 1008.3778; ...

Elliptic genus of K3: definition

- $\mathcal{N} = (4,4)$ SCFT with central charge $c = 6$
- Non-linear σ-model with target space K3
- The model depends on the choice of Einstein metric and B-field (moduli space of theories).

Elliptic genus

$$\phi_{K3}(\tau, z) = \mathrm{Tr}_{RR}\left((-1)^{F+\tilde{F}}\, q^{L_0 - \frac{c}{24}}\, \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}}\, y^{2J_0^3}\right)$$

where $q = e^{2\pi i\tau}$, $y = e^{2\pi i z}$.

J_0^3 is the 3rd comp of (left) $SU(2)$ in $\mathcal{N} = (4,4)$ SC algebra

Elliptic genus of K3: properties

$$\phi_{K3}(\tau, z) = \mathrm{Tr}_{RR}\big((-1)^{F+\tilde{F}}\, q^{L_0-\frac{c}{24}}\, \bar{q}^{\tilde{L}_0-\frac{\tilde{c}}{24}}\, y^{2J_0^3}\big)$$

- Independent of the moduli (metric and B-field)
- Holomorphic in τ and z
- Quasi-periodicity and modular properties:

$$\phi(\tau, z+\ell\tau+\ell') = e^{-2\pi i(\ell^2\tau+2\ell z)}\,\phi(\tau, z) \qquad \ell, \ell' \in \mathbb{Z}$$

$$\phi\Big(\frac{a\tau+b}{c\tau+d}, \frac{z}{c\tau+d}\Big) = e^{2\pi i\frac{cz^2}{c\tau+d}}\,\phi(\tau, z) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Jacobi form of weight 0 and index 1

$\mathcal{N} = 4$ characters and Mathieu representations

Decompose ϕ_{K3} into $\mathcal{N} = 4$ characters

$$\phi_{K3}(\tau, z) = A_{00}\, \mathrm{ch}^{short}_{h=\frac{1}{4},\ell=0}(\tau, z) + \sum_{n=0}^{\infty} A_n\, \mathrm{ch}^{long}_{h=\frac{1}{4}+n,\ell=\frac{1}{2}}(\tau, z)$$

Observation:

A_n are the dimensions of reps H_n of Mathieu group $\mathbb{M}_{24}$

[Eguchi, Ooguri, Tachikawa 1004.0956]

$$H_{00} = \mathbf{23} + \mathbf{1} \qquad H_0 = 2 \cdot \mathbf{1}$$

$$H_1 = \mathbf{45} + \overline{\mathbf{45}} \qquad H_2 = \mathbf{231} + \overline{\mathbf{231}}$$

$$H_3 = \mathbf{770} + \overline{\mathbf{770}} \qquad H_4 = \mathbf{2277} + \overline{\mathbf{2277}}$$

$$H_5 = 2 \cdot \mathbf{5796} \qquad H_6 = 2 \cdot \mathbf{3520} + 2 \cdot \mathbf{10395}$$

Monstrous Moonshine

- Consider $SL(2,\mathbb{Z})$-invariant J-function:

$$J(\tau) = q^{-1} + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

Coefficients are dimension of reps of Monster group F_1 [McKay]

$J(\tau)$ is the partition function of a chiral CFT $V^\natural$ ($c = 24$) with symmetry group F_1 [Frenkel, Lepowski, Meurman '88]

$$J(\tau) = \mathrm{Tr}_{V^\natural}(q^{L_0 - \frac{c}{24}})$$

- McKay-Thompson series [Thompson '79; Conway, Norton '79]:

$$T_g(\tau) = \mathrm{Tr}_{V^\natural}(g\, q^{L_0 - \frac{c}{24}})\,, \qquad g \in F_1$$

  - They depend only on the conjugacy class of g
  - Modular invariant under congruence subgroups of $SL(2,\mathbb{R})$

A Mathieu Moonshine?

$$\phi_g(\tau,z)=\mathrm{Tr}_{RR}\big(g\,(-1)^{F+\tilde F}q^{L_0-\frac{c}{24}}\,\bar q^{\tilde L_0-\frac{\tilde c}{24}}\,y^{2J_0^3}\big),\qquad g\in\mathbb{M}_{24}$$

- 21 different *twining genera*
  [Cheng 1005.5415; Gaberdiel, Hohenegger, R.V. 1006.0221]
- Expected modular properties ($N=ord(g)$):

$$\phi_g\Big(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\Big)=e^{\frac{2\pi ic}{Nh}}\,e^{\frac{2\pi i\,cz^2}{c\tau+d}}\,\phi_g(\tau,z)$$

with $c\equiv 0 \mod N$, for some $h|\gcd(12,N)$.

They are all Jacobi forms under $\Gamma_0(N)$, up to phases
[Gaberdiel, Hohenegger, R.V. 1008.3778; Eguchi, Hikami 1008.4924]

A Mathieu Moonshine?

- Twining genera ϕ_g are based on the "guess" for the first few H_n
- Decomposition of H_n into irreps of $\mathbb{M}_{24}$

$$H_n = \oplus_i h_{n,i} R_i$$

  Once all ϕ_g are known, we can compute multiplicities $h_{n,i}$ for arbitrary n
- By direct computation, $h_{n,i}$ are non-negative integer (check up to $n = 500$)

This is very strong evidence in favour of the conjecture

What's exactly the conjecture?

ϕ_{K3} is an index of σ-models on K3.
Is $\mathbb{M}_{24}$ the symmetry group of any of these?

Symmetries at special points in moduli space:

Symplectic automorphisms of K3

They give rise to some twining genera [David, Jatkar, Sen hep-th/0602254].
They form subgroups of $\mathbb{M}_{23} \subseteq \mathbb{M}_{24}$.

Stringy symmetries

T-duality on $T^4/\mathbb{Z}_2$ at self-dual radius gives a ϕ_g, $g \notin \mathbb{M}_{23}$
[Gaberdiel, Hohenegger, R.V., work in progress]

But it seems unlikely that the whole $\mathbb{M}_{24}$ appears in a single model.

Conclusions and outlook

Summary:

- Elliptic genus of K3 decomposes into $\mathbb{M}_{24}$ representations
- Twining genera have good modular properties

To be understood:

- Is there a theory with $\mathbb{M}_{24}$ symmetry? Some possibilities:
  - A σ-model on K3
  - A different CFT somehow related to σ-models on K3
- Is there an analogue of the genus 0 property of McKay-Thompson series?

References

M.R.Gaberdiel, S.Hohenegger, R.Volpato, 1006.0221 [hep-th]; 1008.3778 [hep-th]; work in progress...