Closed flux tubes and their string description: a lesson by lattice gauge theories.

Andreas Athenodorou*

DESY, Zeuthen

DESY 2010 Theory Conference
Quantum Field Theory: Developments and Perspectives
September 22nd, 2010, DESY, Hamburg.

* With Dr. Barak Bringoltz (Washington)
and Dr. Michael Teper (Oxford) based on:
arXiv:0709.0693, 0812.0334, 0912.3238, 1007.4720
and unpublished results

Overview

1. **Introduction.**
2. **Theoretical Predictions.**
3. **Lattice formulation.**
4. $D = 2 + 1$**.**
   - **Quantum Numbers/Operators.**
   - **Results.**
5. $D = 3 + 1$**.**
   - **Quantum Numbers/Operators.**
   - **Results.**
6. **Conclusions.**

1. Introduction: General

General question:

$\longrightarrow$ What effective string theory describes flux tubes in $SU(N)$ gauge theories?

Two cases:

$\longrightarrow$ Open string

$\longrightarrow$ Closed string (torelon)

During the last decade:

$\longrightarrow$ $3D$, $4D$ with Z_2, Z_4, $SU(N \leq 8)$ (Caselle and collaborators, Gliozzi and collaborators, Kuti and collaborators, Lüscher & Weisz, Majumdar and collaborators, Teper and collaborators)

Recently in $D = 2 + 1$: Nambu-Goto works well for:

$\longrightarrow$ Fundamental Closed String Spectrum.(A.A, B. Bringoltz, M. Teper)[arXiv:hep-lat/0709.0693]

First attempt to extract the $D = 3 + 1$ torelon spectrum in $SU(3)$:

K. J. Juge, J. Kuti, F. Maresca, C. Morningstar and M. Peardon in [arXiv:hep-lat/0309180]

1. Introduction: General

General question:

→ What effective string theory describes flux tubes in $SU(N)$ gauge theories?

Two cases:

→ Open string

→ **Closed string** (torelon) ← in $D = 2 + 1$ & $D = 3 + 1$

During the last decade:

→ $3D$, $4D$ with Z_2, Z_4, $SU(N \leq 8)$ (Caselle and collaborators, Gliozzi and collaborators, Kuti and collaborators, Lüscher & Weisz, Majumdar and collaborators, Teper and collaborators)

Recently in $D = 2 + 1$: Nambu-Goto works well for:

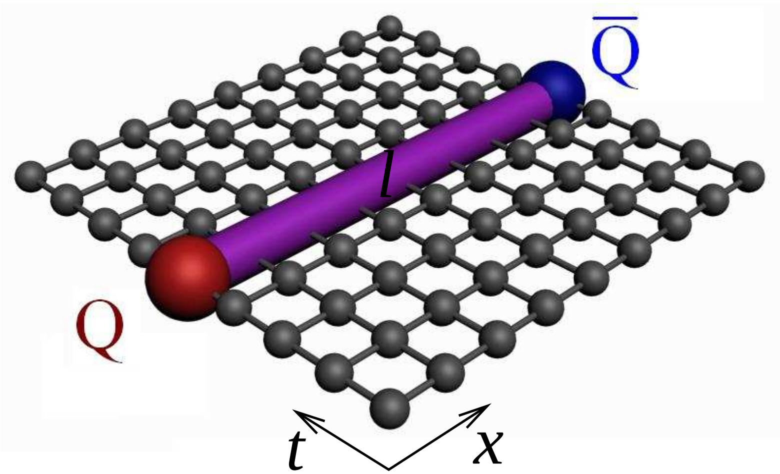
→ Fundamental Closed String Spectrum.(A.A, B. Bringoltz, M. Teper)[arXiv:hep-lat/0709.0693]

First attempt to extract the $D = 3 + 1$ torelon spectrum in $SU(3)$:

K. J. Juge, J. Kuti, F. Maresca, C. Morningstar and M. Peardon in [arXiv:hep-lat/0309180]

1. Introduction: Open-Closed (torelons) flux tubes

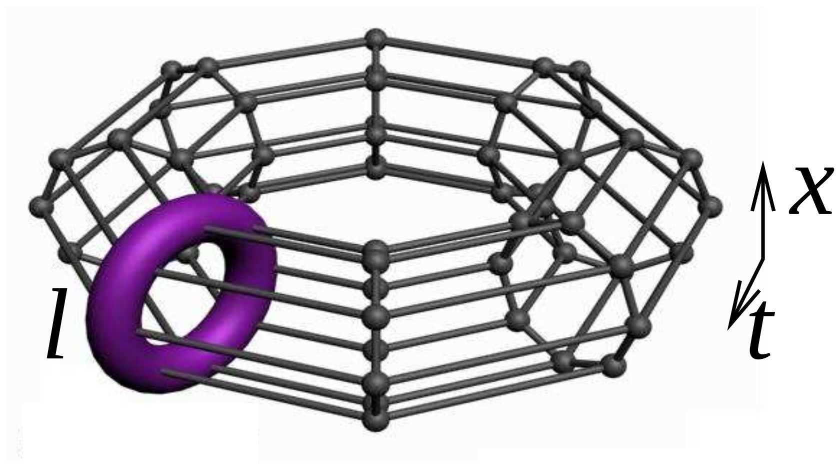
Open flux tube



$$\Phi(l,t) = \psi^{\dagger}(0,t)U(0,l;t)\psi(l,t)$$

Periodic B.C

Closed flux tube



$$\Phi(l,t) = \mathrm{Tr}U(l;t)$$

2. Theoretical Predictions: Effective String Theory.

The energy (mass) of a closed flux tube is expected to be described as:

$$E_n \stackrel{l\to\infty}{=} \sigma l \qquad \text{linear confinement}$$

2. Theoretical Predictions: Effective String Theory.

The energy (mass) of a closed flux tube is expected to be described as:

$$E_n \stackrel{l\to\infty}{=} \sigma l \qquad \text{linear confinement}$$

$$+ \quad \frac{4\pi}{l}\left(n - \frac{D-2}{24}\right) \qquad \text{Lüscher 1980, Polchinski\&Strominger 1991}$$

2. Theoretical Predictions: Effective String Theory.

The energy (mass) of a closed flux tube is expected to be described as:

$$E_n \stackrel{l\to\infty}{=} \sigma l$$ linear confinement

$$+ \frac{4\pi}{l}\left(n - \frac{D-2}{24}\right)$$ Lüscher 1980, Polchinski&Strominger 1991

$$- \frac{8\pi^2}{\sigma l^3}\left(n - \frac{D-2}{24}\right)^2$$ Lüscher&Weisz 2004, Drummond 2004

2. Theoretical Predictions: Effective String Theory.

The energy (mass) of a closed flux tube is expected to be described as:

$$E_n \stackrel{l\to\infty}{=} \sigma l \qquad \text{linear confinement}$$

$$+ \frac{4\pi}{l}\left(n - \frac{D-2}{24}\right) \qquad \text{Lüscher 1980, Polchinski\&Strominger 1991}$$

$$- \frac{8\pi^2}{\sigma l^3}\left(n - \frac{D-2}{24}\right)^2 \qquad \text{Lüscher\&Weisz 2004, Drummond 2004}$$

$$+ \frac{32\pi^3}{\sigma^2 l^5}\left(n - \frac{D-2}{24}\right)^3 \qquad \text{Aharony\&Karzbrun 2009}$$

2. Theoretical Predictions: Effective String Theory.

The energy (mass) of a closed flux tube is expected to be described as:

$$
\begin{aligned}
E_n \stackrel{l\to\infty}{=} \quad & \sigma l && \text{linear confinement} \\
+ \quad & \frac{4\pi}{l}\left(n - \frac{D-2}{24}\right) && \text{Lüscher 1980, Polchinski\&Strominger 1991} \\
- \quad & \frac{8\pi^2}{\sigma l^3}\left(n - \frac{D-2}{24}\right)^2 && \text{Lüscher\&Weisz 2004, Drummond 2004} \\
+ \quad & \frac{32\pi^3}{\sigma^2 l^5}\left(n - \frac{D-2}{24}\right)^3 && \text{Aharony\&Karzbrun 2009} \\
+ \quad & \cdots + \mathcal{O}\left(\frac{1}{l^\infty}\right) = E_{N.G} && \text{Dass\&collaborators 2009}
\end{aligned}
$$

2. Theoretical Expectations: Nambu-Goto String

→ **Spectrum:**

$$E^2_{N_L,N_R,q,w} = (\sigma l w)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{D-2}{24}\right) + \left(\frac{2\pi q}{l}\right)^2 .$$

→ **Described by:**

1. The winding number w (w=1),
2. The winding momentum $p_{\|} = 2\pi q/l$ with $q = 0, \pm 1, \pm 2, \ldots$.
3. The transverse momentum $p_\perp$ ($p_\perp = 0$),
4. N_L and N_R connected through the relation: $N_L - N_R = qw$.

$$N_L = \sum_{k>0} \sum_{n_L(k)>0} n_L(k) k \quad \text{and} \quad N_R = \sum_{k'>0} \sum_{n_R(k')>0} n_R(k') k'$$

→ **Construction of states:**

$$(\alpha^{i_1}_{-k_1})^{n_L(k_1)} \ldots (\alpha^{i_{m_L}}_{-k_{m_L}})^{n_R(k_{m_L})} (\bar{\alpha}^{i'_1}_{-k'_1})^{n_R(k'_1)} \ldots (\bar{\alpha}^{i'_{m_R}}_{-k'_{m_R}})^{n_R(k'_{m_R})} \mid 0\rangle$$

($i = 1, \ldots, D-2$)

Example: $\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1} \mid 0\rangle \rightarrow N_L = 3, N_R = 1,\ q = 2\ (w = 1)$

2. Theoretical Expectations: Nambu-Goto String

→ **Spectrum:**

$$E^2_{N_L,N_R,q,w} = (\sigma l w)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{D-2}{24}\right) + \left(\frac{2\pi q}{l}\right)^2 .$$

→ **Described by:**

1. The winding number w (w=1),
2. The winding momentum $p_{\|} = 2\pi q/l$ with $q = 0, \pm 1, \pm 2, \ldots$.
3. The transverse momentum $p_\perp$ ($p_\perp = 0$),
4. N_L and N_R connected through the relation: $N_L - N_R = qw$.

$$N_L = \sum_{k>0} \sum_{n_L(k)>0} n_L(k) k \quad \text{and} \quad N_R = \sum_{k'>0} \sum_{n_R(k')>0} n_R(k') k'$$

→ **Construction of states:**

$$(\alpha^{i_1}_{-k_1})^{n_L(k_1)} \ldots (\alpha^{i_{m_L}}_{-k_{m_L}})^{n_R(k_{m_L})} (\bar{\alpha}^{i'_1}_{-k'_1})^{n_R(k'_1)} \ldots (\bar{\alpha}^{i'_{m_R}}_{-k'_{m_R}})^{n_R(k'_{m_R})} \mid 0\rangle$$

$(i = 1, \ldots, D-2)$

Example: $\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1} \mid 0\rangle \rightarrow N_L = 3, N_R = 1,\ q = 2\ (w = 1)$

2. Theoretical Expectations: Nambu-Goto String

→ **Spectrum:**

$$E^2_{N_L,N_R,q,w} = (\sigma l w)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{D-2}{24}\right) + \left(\frac{2\pi q}{l}\right)^2 .$$

→ **Described by:**

1. The winding number w (w=1),
2. The winding momentum $p_{\|} = 2\pi q/l$ with $q = 0, \pm 1, \pm 2, \ldots$.
3. The transverse momentum $p_\perp$ ($p_\perp = 0$),
4. N_L and N_R connected through the relation: $N_L - N_R = qw$.

$$N_L = \sum_{k>0} \sum_{n_L(k)>0} n_L(k)k \quad \text{and} \quad N_R = \sum_{k'>0} \sum_{n_R(k')>0} n_R(k')k'$$

→ **Construction of states:**

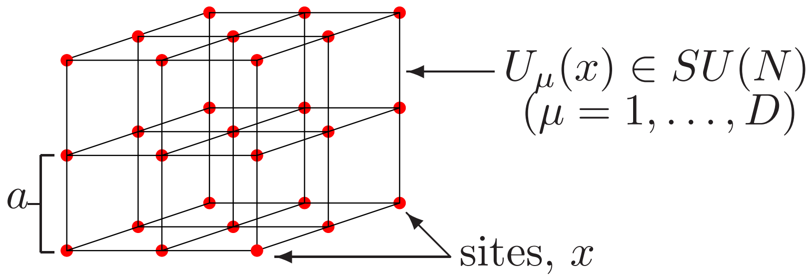
$$(\alpha^{i_1}_{-k_1})^{n_L(k_1)} \ldots (\alpha^{i_{m_L}}_{-k_{m_L}})^{n_R(k_{m_L})} (\bar{\alpha}^{i'_1}_{-k'_1})^{n_R(k'_1)} \ldots (\bar{\alpha}^{i'_{m_R}}_{-k'_{m_R}})^{n_R(k'_{m_R})} \mid 0\rangle$$

$(i = 1, \ldots, D-2)$

Example: $\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1} \mid 0\rangle \rightarrow N_L = 3, N_R = 1,\ q = 2\ (w = 1)$

3. Lattice Calculation: Lattice Setup

- Define the gauge theory on a $D = 3, 4$ discretized periodic Euclidean space-time lattice with $L_{\parallel} \times L_{\perp} \times L_T$ and $L_{\parallel} \times L_{\perp} \times L_{\perp} \times L_T$ sites.



- We use the standard Wilson Action:

$$
\begin{aligned}
S_L &= \beta \sum_p \{1 - \frac{1}{N_c} ReTrU_p\} \\
\beta &= \frac{2N_c}{ag^2}(D = 2 + 1), \frac{2N_c}{g^2}(D = 3 + 1) \\
\lambda &= g^2 N
\end{aligned}
$$

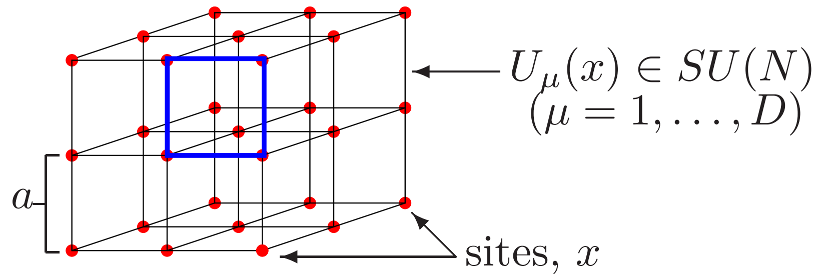
- Energies can be calculated using the correlation functions of specific operators:

$$
\begin{aligned}
C(t) &= \langle \Phi^{\dagger}(t)\Phi(0)\rangle = \langle \Phi^{\dagger}(0)e^{-Ht}\Phi(0)\rangle \\
&= |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t} + \sum_{n=1} |\langle n|\Phi(0)|vac\rangle|^2 e^{-E_n t} \xrightarrow{t\to\infty} |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t}
\end{aligned}
$$

- Excitations can be calculated using the variational technique

3. Lattice Calculation: Lattice Setup

- Define the gauge theory on a $D = 3, 4$ discretized periodic Euclidean space-time lattice with $L_{\parallel} \times L_{\perp} \times L_T$ and $L_{\parallel} \times L_{\perp} \times L_{\perp} \times L_T$ sites.



- We use the standard Wilson Action:

$$S_L = \beta \sum_p \{1 - \frac{1}{N_c} ReTrU_p\}$$

$$\beta = \frac{2N_c}{ag^2}(D = 2+1), \frac{2N_c}{g^2}(D = 3+1)$$

$$\lambda = g^2 N$$

- Energies can be calculated using the correlation functions of specific operators:

$$C(t) = \langle \Phi^\dagger(t)\Phi(0)\rangle = \langle \Phi^\dagger(0)e^{-Ht}\Phi(0)\rangle$$

$$= |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t} + \sum_{n=1} |\langle n|\Phi(0)|vac\rangle|^2 e^{-E_n t} \xrightarrow{t\to\infty} |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t}$$

- Excitations can be calculated using the variational technique.

3. Lattice Calculation: Variational Technique

- Construct a large basis of Operators (Polyakov loops) $\Phi_i : i = 1, 2, \ldots$ described by the right quantum numbers

- Calculate the correlation function (Matrix) $C_{ij}(t) = \langle \Phi_i^\dagger(t) \Phi_j(0) \rangle$

- Diagonalise the matrix $C^{-1}(t = 0) C(t = a)$

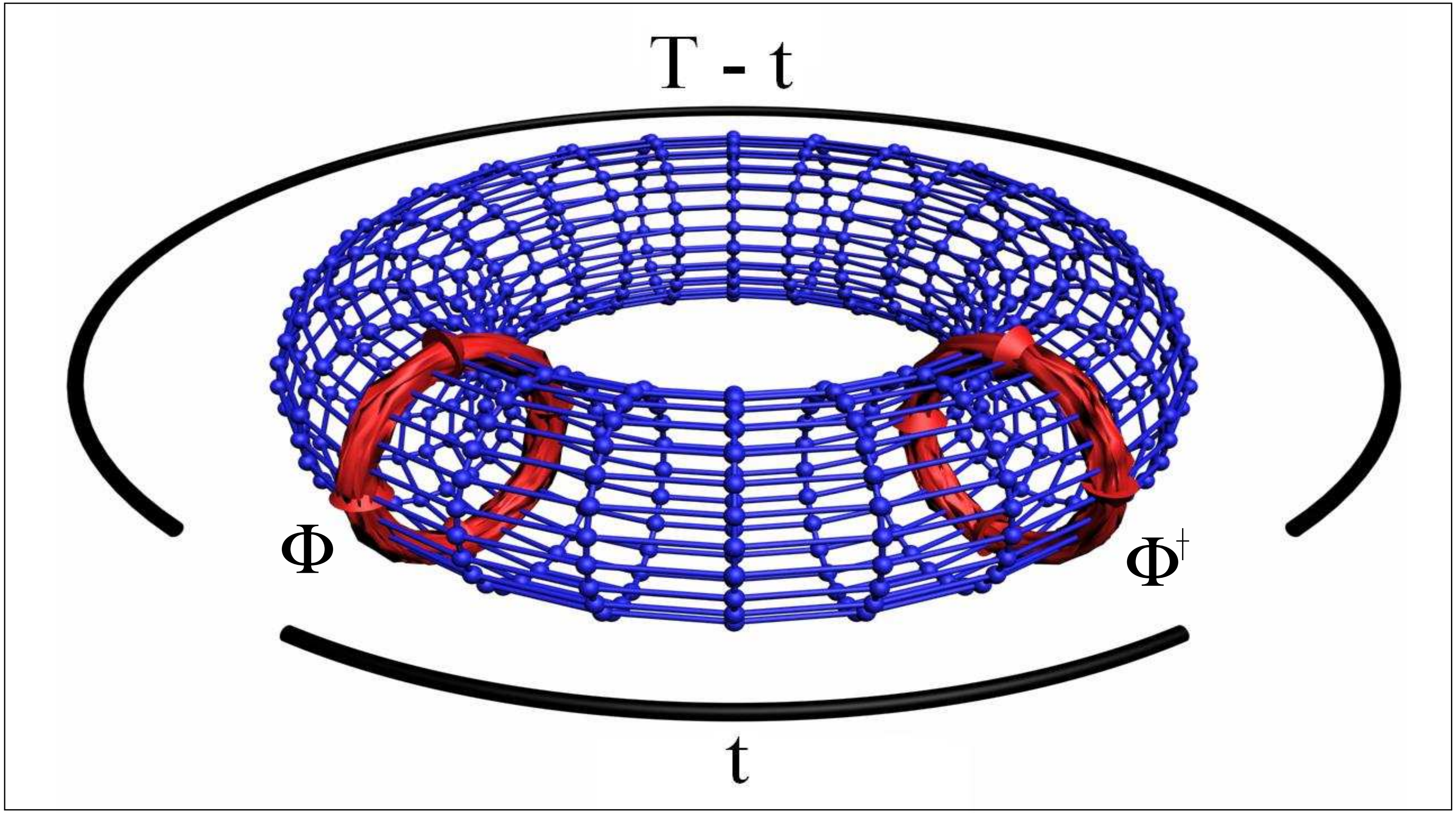
- Extract the eigenvectors

- Extract the correlator for each state ($\sim \mathrm{e}^{-E_n t}$)

- By fitting the results, we extract the mass (energy) for each state.

3. Lattice Calculation: Correlation Function

Pictorialisation of the Correlation Function



4. D=2+1

4. D=2+1: Lattice Calculation

→ **We define our theory on a $3D$ Euclidean lattice with volume:**

$$L_{\|} \times L_{\perp_1} \times L_T.$$

→ **We use the standard Wilson Action:**

$$S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \mathrm{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{ag^2}$$

→ **Monte Carlo simulations:**

- $SU(3)$ at $\beta = 21$, $a\sqrt{\sigma} \simeq 0.174$
- $SU(3)$ at $\beta = 40$, $a\sqrt{\sigma} \simeq 0.087$
- $SU(4)$ at $\beta = 50$, $a\sqrt{\sigma} \simeq 0.131$
- $SU(5)$ at $\beta = 80$, $a\sqrt{\sigma} \simeq 0.130$
- $SU(6)$ at $\beta = 80$, $a\sqrt{\sigma} \simeq 0.172$
- $SU(6)$ at $\beta = 171$, $a\sqrt{\sigma} \simeq 0.086$

4. D=2+1: Lattice Calculation

→ **We define our theory on a $3D$ Euclidean lattice with volume:**

$$L_{\|} \times L_{\perp_1} \times L_T .$$

→ **We use the standard Wilson Action:**

$$S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \mathrm{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{ag^2}$$

→ **Monte Carlo simulations:**

- $SU(3)$ at $\beta = 21$, $a\sqrt{\sigma} \simeq 0.174$ [arXiv:0709.0693]
- $SU(3)$ at $\beta = 40$, $a\sqrt{\sigma} \simeq 0.087$ [arXiv:0709.0693]
- $SU(4)$ at $\beta = 50$, $a\sqrt{\sigma} \simeq 0.131$ [arXiv:0812.0334]
- $SU(5)$ at $\beta = 80$, $a\sqrt{\sigma} \simeq 0.130$ [arXiv:0812.0334]
- $SU(6)$ at $\beta = 80$, $a\sqrt{\sigma} \simeq 0.172$ [arXiv:0709.0693]
- $SU(6)$ at $\beta = 171$, $a\sqrt{\sigma} \simeq 0.086$

4. D=2+1: Lattice Calculation

→ **We define our theory on a $3D$ Euclidean lattice with volume:**

$$L_{\|} \times L_{\perp_1} \times L_T .$$

→ **We use the standard Wilson Action:**

$$S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \mathrm{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{ag^2}$$

→ **Monte Carlo simulations:**

- $SU(3)$ at $\beta = 21$, $a\sqrt{\sigma} \simeq 0.174$ [arXiv:0709.0693]
- $SU(3)$ at $\beta = 40$, $a\sqrt{\sigma} \simeq 0.087$ [arXiv:0709.0693]
- $SU(4)$ at $\beta = 50$, $a\sqrt{\sigma} \simeq 0.131$ [arXiv:0812.0334]
- $SU(5)$ at $\beta = 80$, $a\sqrt{\sigma} \simeq 0.130$ [arXiv:0812.0334]
- $SU(6)$ at $\beta = 80$, $a\sqrt{\sigma} \simeq 0.172$ [arXiv:0709.0693]
- **SU(6) at $\beta = 171$, a$\sqrt{\sigma} \simeq$ 0.086**

4. D=2+1: Symmetries

→ **Flux tube:**

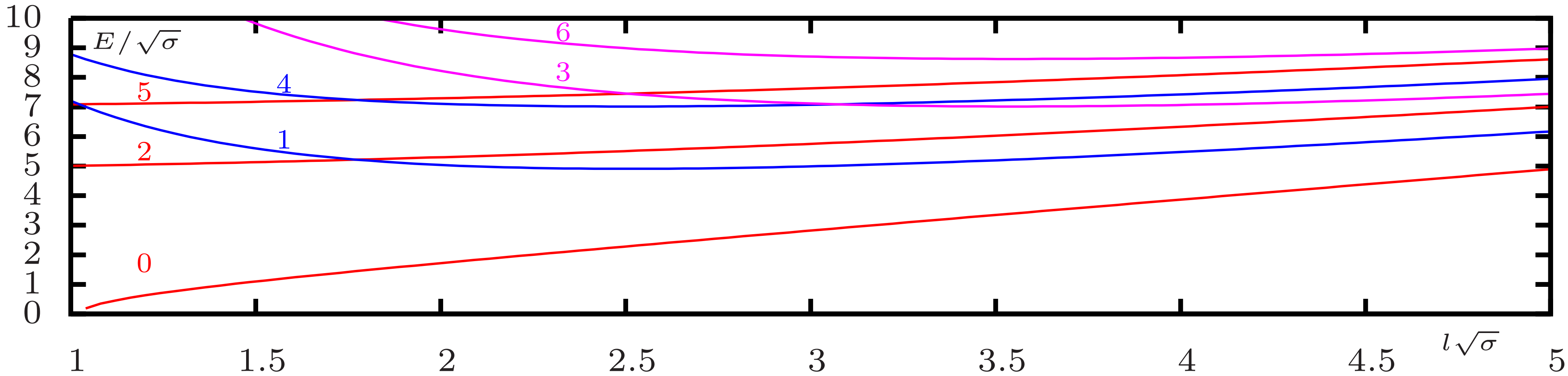
- Lattice Symmetry of Rotations in $D = 2 + 1 \rightarrow$ Reflections on transverse line
- States are described by the quantum number of Parity $P = \pm$
- Need to construct operators trasforming irreducibly under P

→ **Bosonic String:**

- String states can be characterized as irreducible representations of $SO(D-2)$.
- In $D = 2 + 1$ this becomes the transverse parity ($P = \pm$).
- Under Parity: $\alpha_{-k} \overset{P}{\longleftrightarrow} -\alpha_{-k}$ and $\bar{\alpha}_{-k} \overset{P}{\longleftrightarrow} -\bar{\alpha}_{-k}$
- Even number of phonons $\rightarrow P = +$ $(\alpha_{-2}\bar{\alpha}_{-1} \mid 0\rangle)$
- Odd number of phonons $\rightarrow P = -$ $(\alpha_{-1} \mid 0\rangle)$

4. D=2+1: Symmetries: Bosonic String

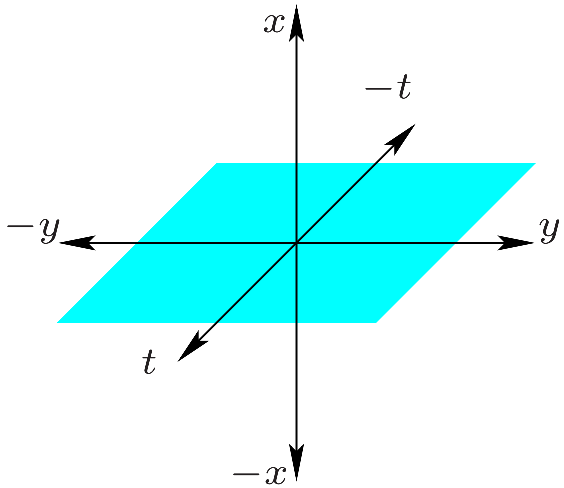
The seven lowest $(q = 0, 1, 2)$ NG energy levels for the $w = 1$ closed string



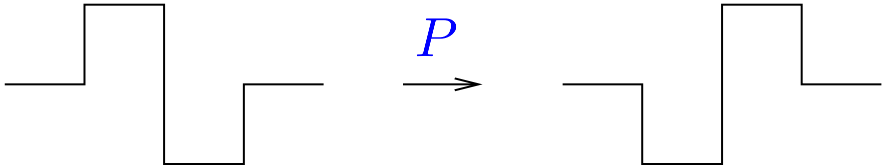
level	N_L	N_R	q	$P=+$	$P=-$
0	0	0	0	$\|0\rangle$	
1	1	0	1		$\alpha_{-1}\|0\rangle$
2	1	1	0	$\alpha_{-1}\bar{\alpha}_{-1}\|0\rangle$	
3	2	0	2	$\alpha_{-1}\alpha_{-1}\|0\rangle$	$\alpha_{-2}\|0\rangle$
4	2	1	1	$\alpha_{-2}\bar{\alpha}_{-1}\|0\rangle$	$\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1}\|0\rangle$
5	2	2	0	$\alpha_{-2}\bar{\alpha}_{-2}\|0\rangle, \quad \alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1}\bar{\alpha}_{-1}\|0\rangle$	$\alpha_{-2}\bar{\alpha}_{-1}\bar{\alpha}_{-1}\|0\rangle, \quad \alpha_{-1}\alpha_{-1}\bar{\alpha}_{-2}\|0\rangle$
6	3	1	2	$\alpha_{-3}\bar{\alpha}_{-1}\|0\rangle, \quad \alpha_{-1}\alpha_{-1}\alpha_{-1}\bar{\alpha}_{-1}\|0\rangle$	$\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1}\|0\rangle$

4. D=2+1: Quantum Numbers: Parity

Parity reflection plane



Example:



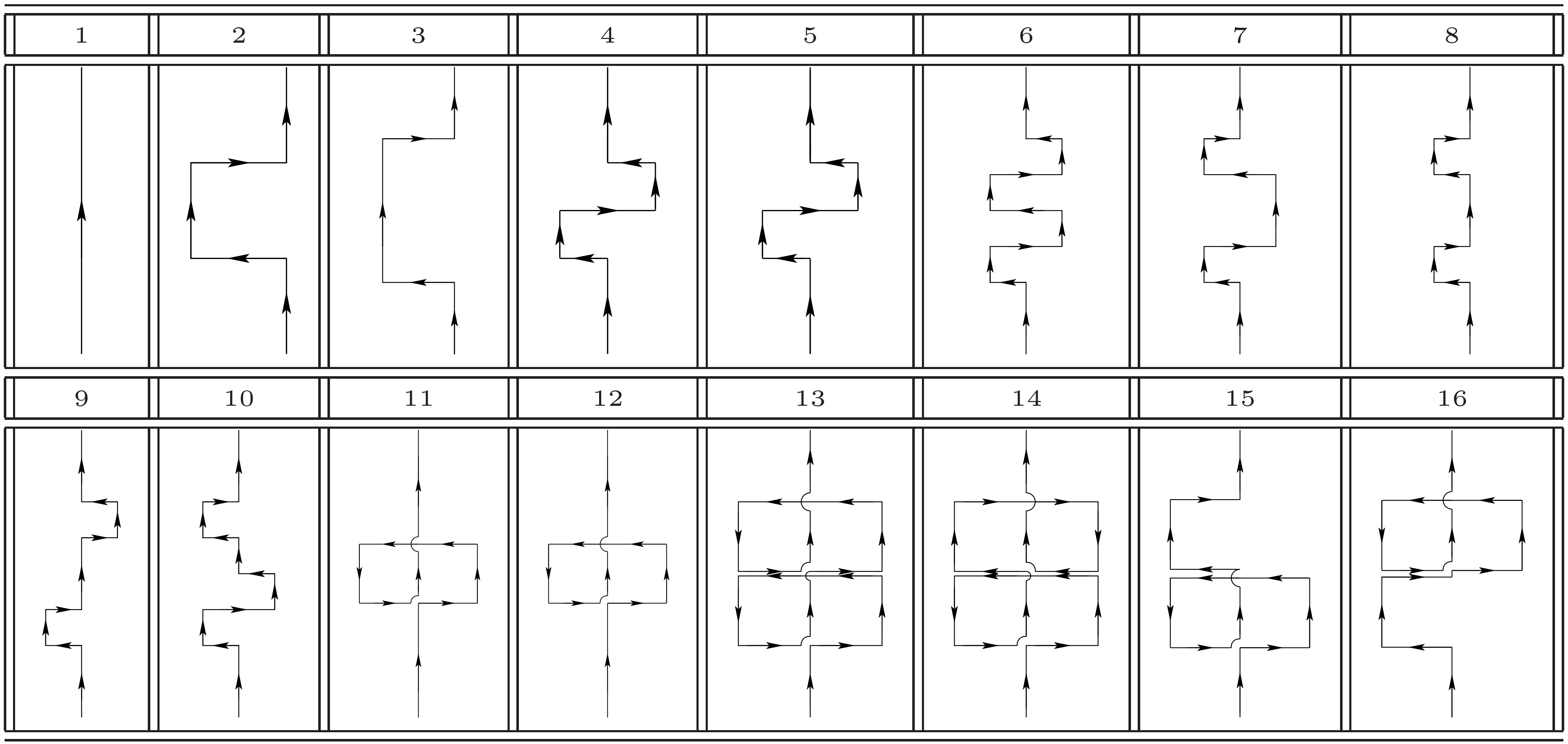
We can project onto $P = \pm$ using operators such as:

$$\Phi(q_{||}, P) = \frac{1}{L_{||} L_{\perp}} \sum_{x_{||}, x_{\perp}} \{ \mathrm{Tr}\ \{ \quad \} \pm \mathrm{Tr} \{ \quad \} \} \ \mathrm{e}^{\mathrm{i}2\pi \mathrm{q}_{||} \mathrm{x}_{||} / \mathrm{L}_{||}}$$

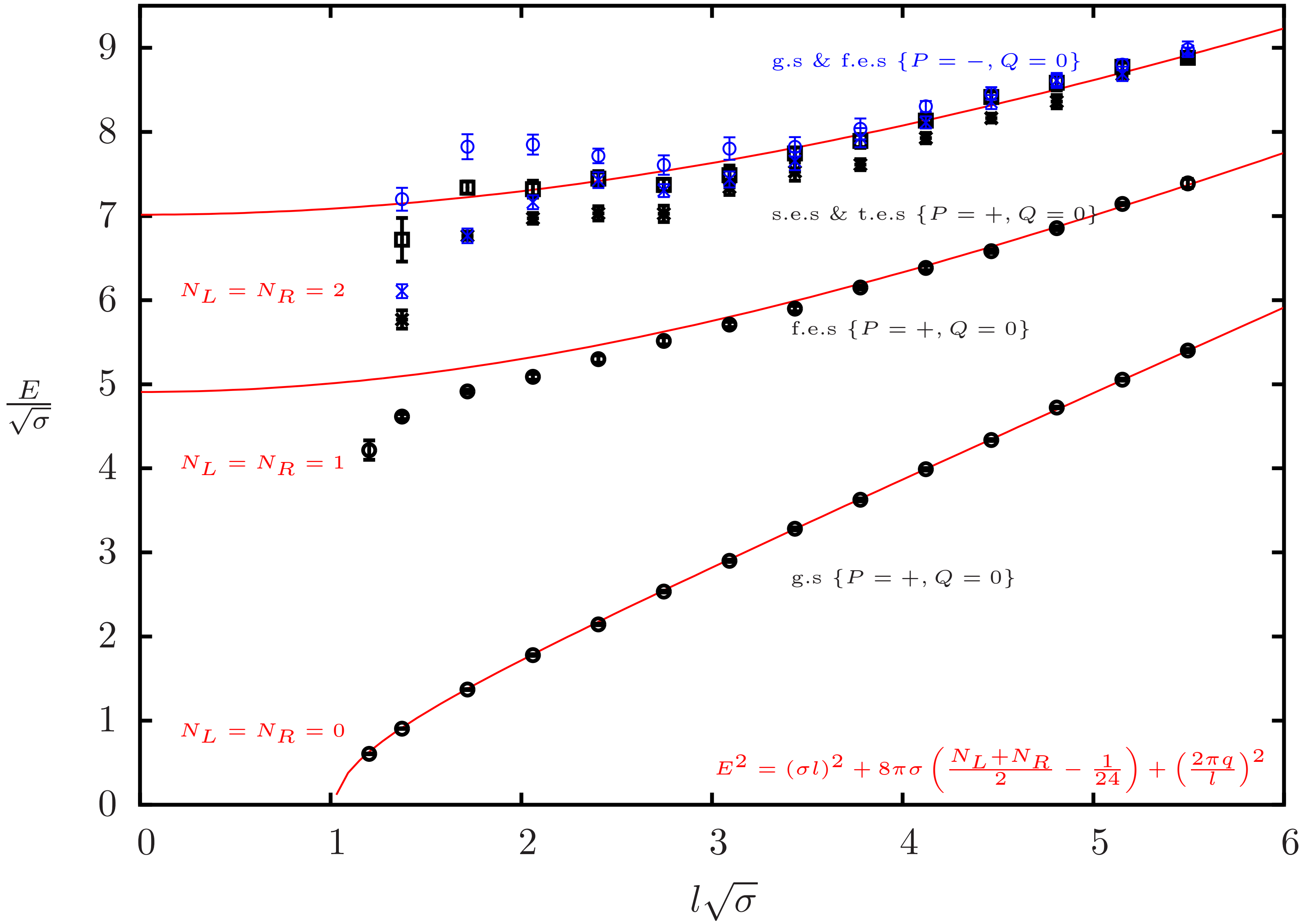
4. D=2+1: Transverse deformations

$\longrightarrow \sim 200$ operators

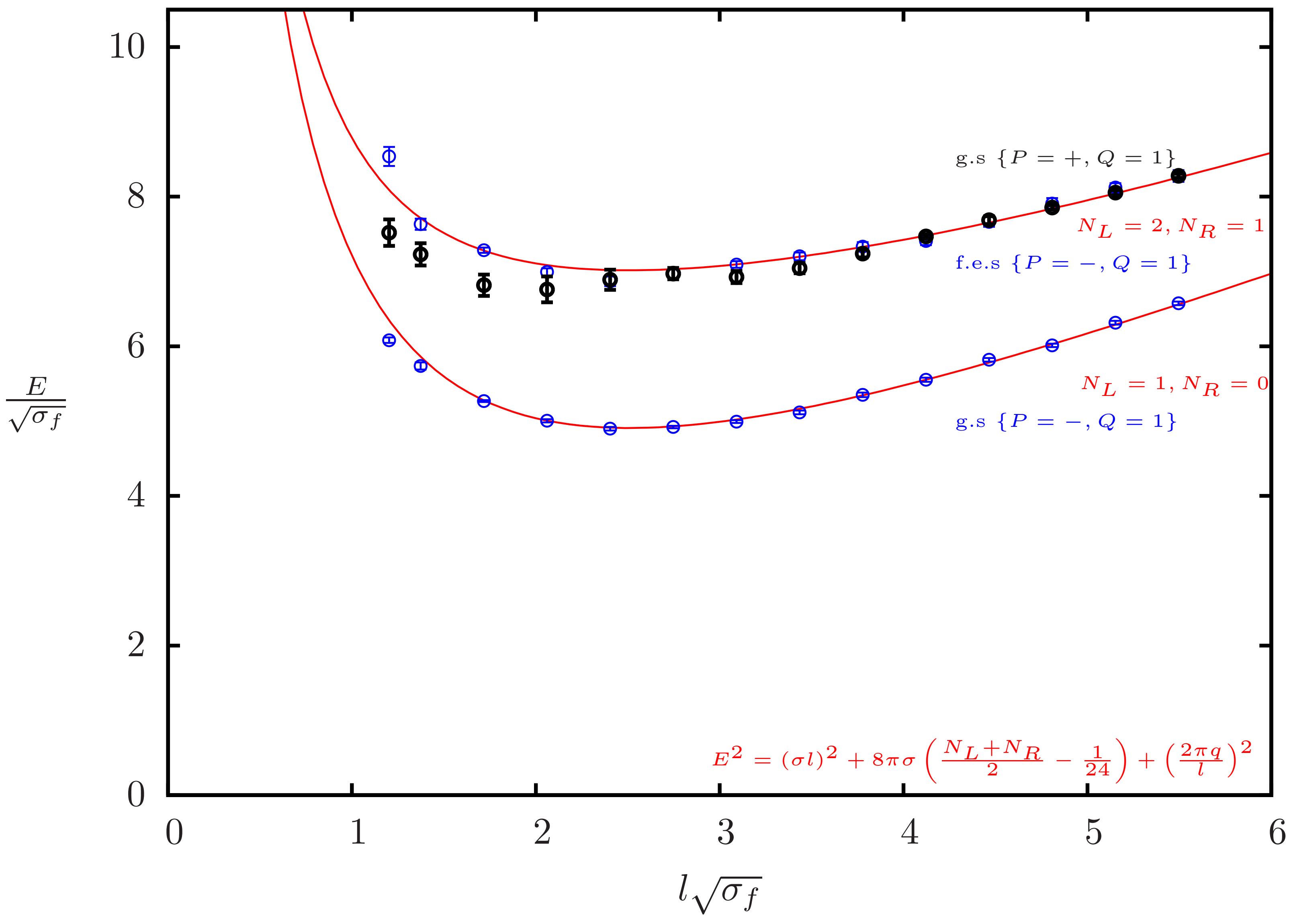
Transverse deformations in 2 spatial dimensions:



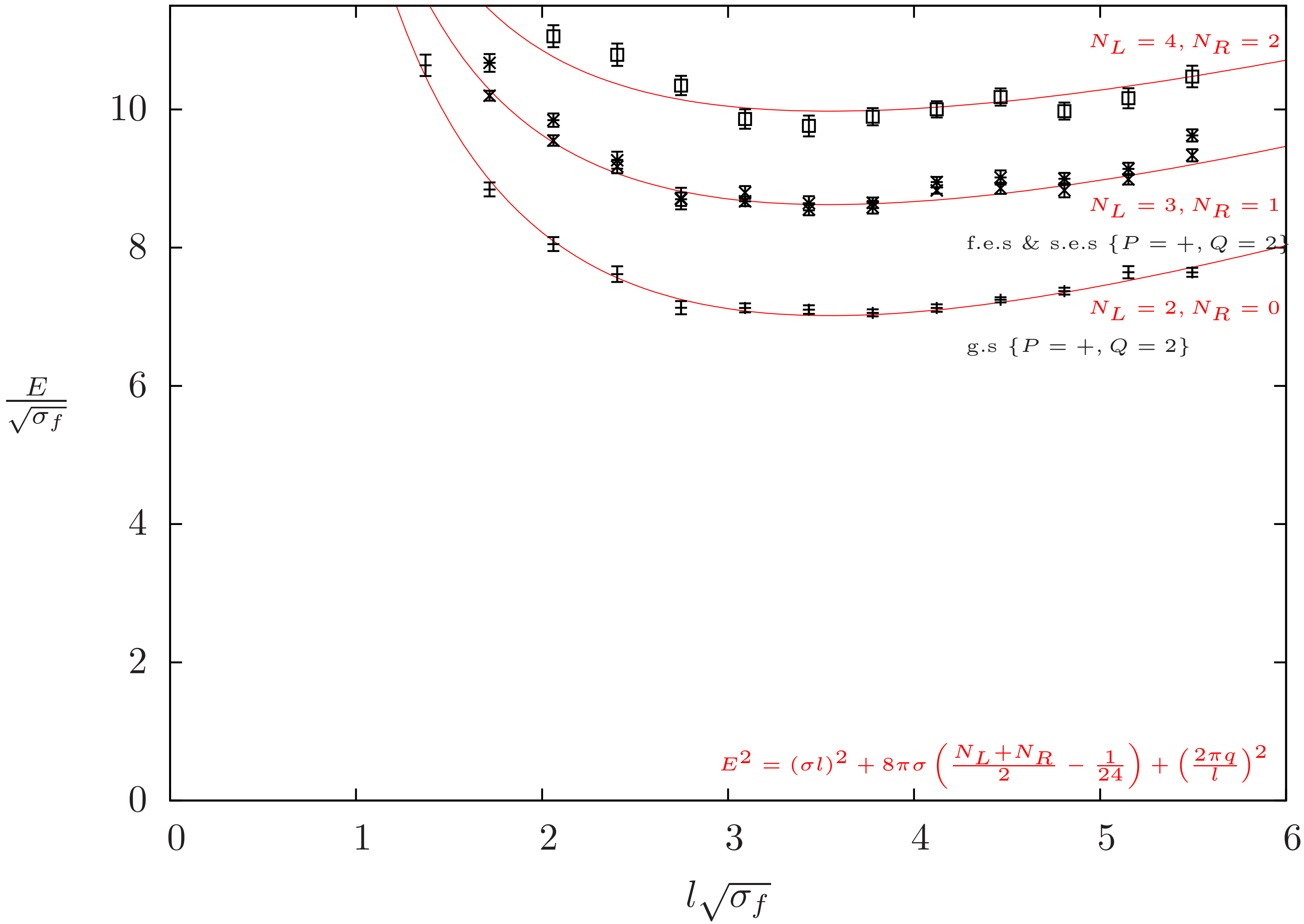
4. D=2+1 Results: $q = 0$, $P = \pm$



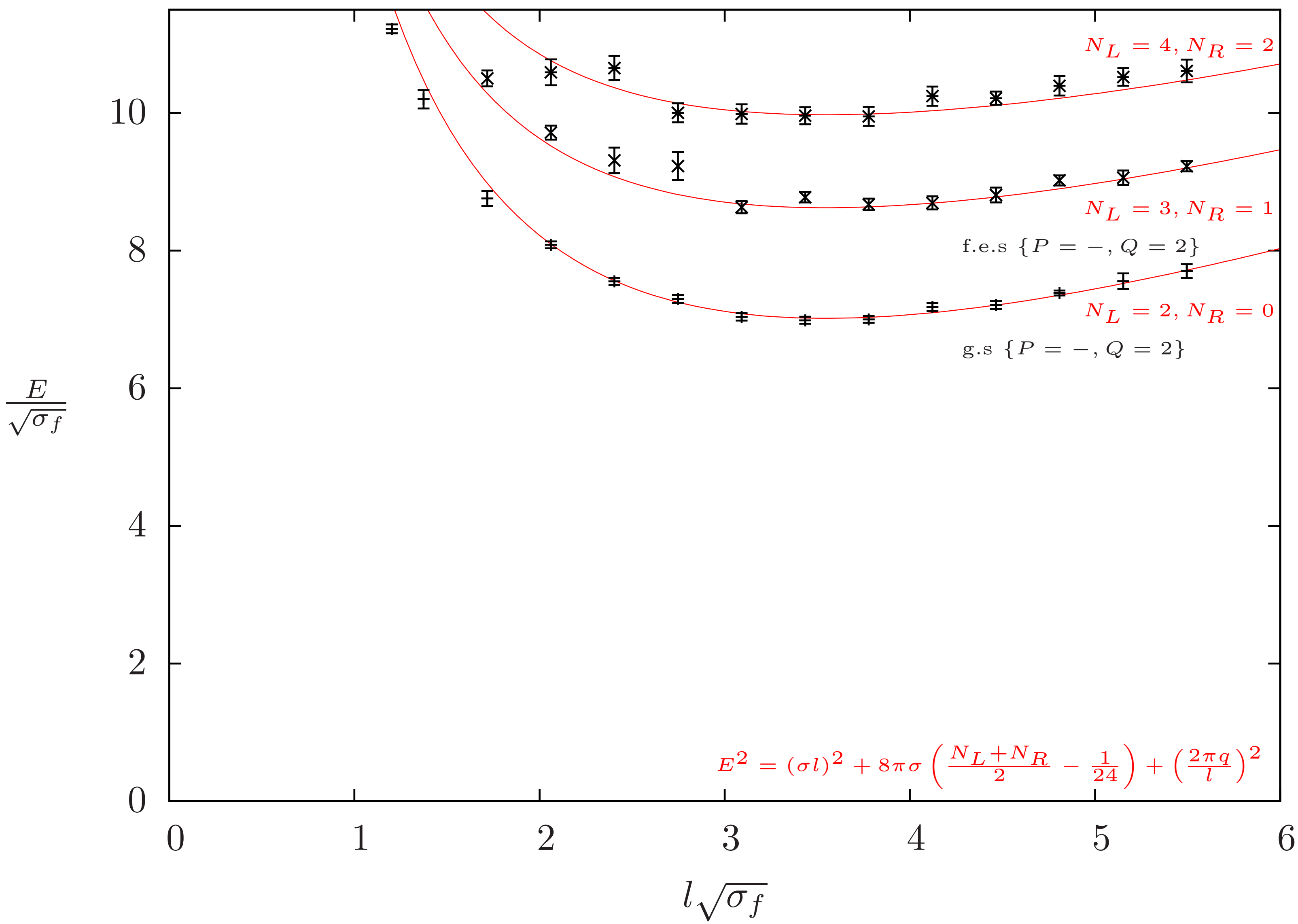
4. D=2+1 Results: $q = 1,\ P = \pm$



4. D=2+1 Results: $q = 2$, $P = +$



4. D=2+1 Results: $q = 2,\ P = -$



5. D=3+1

5. D=3+1: Lattice Calculation

→ **We define our theory on a $4D$ Euclidean lattice with volume:**

$$L_{\|} \times L_{\perp_1} \times L_{\perp_2} \times L_T.$$

→ **We use the standard Wilson Action:**

$$S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \mathrm{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{g^2}$$

→ **Monte Carlo simulations:**

- First: $SU(3)$, $\beta = 6.0625$ with $a\sqrt{\sigma_f} \simeq 0.195$ ($a \simeq 0.09$fm).
- $a \to 0$: $SU(3)$, $\beta = 6.3380$ with $a\sqrt{\sigma_f} \simeq 0.129$ ($a \simeq 0.06$fm).
- $N \to \infty$: $SU(5)$, $\beta = 17.630$ with $a\sqrt{\sigma_f} \simeq 0.197$ ($a \simeq 0.09$fm).

→ **Our Approach:**

- Create a large basis of operators $\Phi_{i,q,J,P_{\mathcal{R}},P_{\mathcal{P}}} : i = 1, 2..., N_O$, ($\sim 700$).
- Calculate the correlation matrix $C_{ij,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}(t) = \langle \Phi^{\dagger}_{i,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}(t) \Phi_{j,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}(0) \rangle$
- Use the variational technique to extract correlators of different states.

5. D=3+1: Lattice Calculation

→ **We define our theory on a $4D$ Euclidean lattice with volume:**

$$L_{\|} \times L_{\perp_1} \times L_{\perp_2} \times L_T .$$

→ **We use the standard Wilson Action:**

$$S_L = \beta \sum_p \left\{ 1 - \frac{1}{N} \mathrm{ReTr} U_p \right\} \quad \text{with} \quad \beta = \frac{2N}{g^2}$$

→ **Monte Carlo simulations:**

- First: $SU(3)$, $\beta = 6.0625$ with $a\sqrt{\sigma_f} \simeq 0.195$ ($a \simeq 0.09$fm).
- $a \to 0$: $SU(3)$, $\beta = 6.3380$ with $a\sqrt{\sigma_f} \simeq 0.129$ ($a \simeq 0.06$fm).
- $N \to \infty$: $SU(5)$, $\beta = 17.630$ with $a\sqrt{\sigma_f} \simeq 0.197$ ($a \simeq 0.09$fm).

→ **Our Approach:**

- Create a large basis of operators $\Phi_{i,q,J,P_{\mathcal{R}},P_{\mathcal{P}}} : i = 1, 2..., N_O$, ($\sim 700$).
- Calculate the correlation matrix $C_{ij,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}(t) = \langle \Phi^{\dagger}_{i,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}(t) \Phi_{j,q,J,P_{\mathcal{R}},P_{\mathcal{P}}}(0) \rangle$
- Use the variational technique to extract correlators of different states.

5. D=3+1: Symmetries: Flux-Tubes

More complicated structure than $D = 2 + 1$:

$\rightarrow$ Lattice Symmetry of Rotations about the string axis.

$\rightarrow$ $C_{4\nu} \bigotimes Z(\mathcal{R})$ for zero longitudinal momentum.

- Rotations of $\pi/2 \rightarrow$ angular momentum J
- Reflections in orthogonal plane ($\mathcal{P}$-Parity)
- Reflections about the mid-point on the principal axis ($\mathcal{R}$-Parity)

$\rightarrow$ 10 irreducible representations $\equiv$ 10 correlation matrices

$\rightarrow$ $C_{4\nu}$ for non-zero longitudinal momentum.

- Rotations of $\pi/2 \rightarrow$ angular momentum J
- Reflections in orthogonal plane ($\mathcal{P}$-Parity)

$\rightarrow$ 5 irreducible representations $\equiv$ 5 correlation matrices

$\rightarrow A_1 \equiv (\mid J \mid = 0, 4, ..., 4N,\ P_{\mathcal{P}} = +),\ A_2 \equiv (\mid J \mid = 0, 4, ..., 4N,\ P_{\mathcal{P}} = -)$

$\rightarrow E \equiv (\mid J \mid = 1, 3, ..., \pm 2N + 1)$

$\rightarrow B_1 \equiv (\mid J \mid = 2, 6, ..., 4N + 2,\ P_{\mathcal{P}} = +),\ B_2 \equiv (\mid J \mid = 2, 6, ..., 4N + 2,\ P_{\mathcal{P}} = -)$

5. D=3+1: Symmetries: Bosonic String

Two transverse directions:

→ Define α^{+}_{-k} and α^{-}_{-k} as $(x, y$ are the transverse directions):

- $\alpha^{+}_{-k} = \alpha^{x}_{-k} + i\alpha^{y}_{-k}$
- $\alpha^{-}_{-k} = \alpha^{x}_{-k} - i\alpha^{y}_{-k}$

→ Spin J.

- $J = \mid \#(+) - \#(-) \mid$

→ $\mathcal{P}$-Parity

- Under $\mathcal{P}$-Parity: $\alpha^{+}_{-k} \overset{P_{\mathcal{P}}}{\longleftrightarrow} \alpha^{-}_{-k}$ & $\bar{\alpha}^{+}_{-k} \overset{P_{\mathcal{P}}}{\longleftrightarrow} \bar{\alpha}^{-}_{-k}$

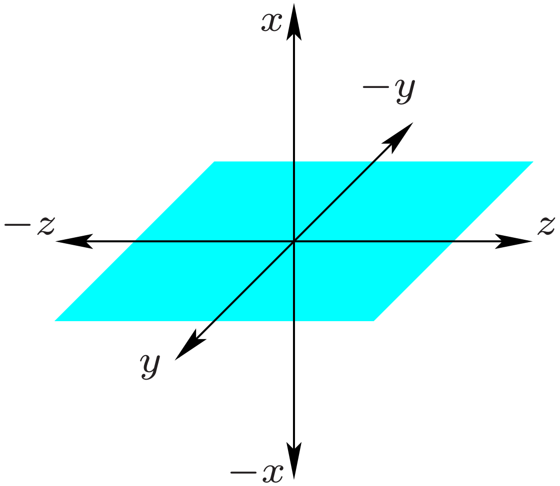
→ $\mathcal{R}$-Parity

- Under $\mathcal{R}$-Parity: $\alpha^{\pm}_{-k} \overset{P_{\mathcal{R}}}{\longleftrightarrow} \bar{\alpha}^{\pm}_{-k}$

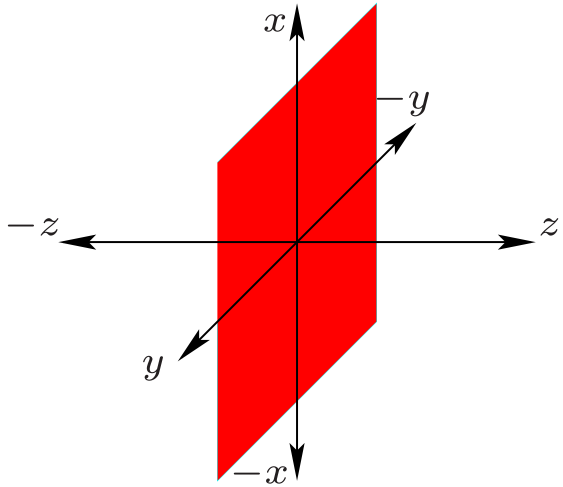
- Example: $(\alpha^{+}_{-1}\bar{\alpha}^{+}_{-1} \pm \alpha^{-}_{-1}\bar{\alpha}^{-}_{-1}) \mid 0\rangle$
  - $J = 2$
  - $P_{\mathcal{P}} = \pm$
  - $P_{\mathcal{R}} = +$

5. D=3+1: Quantum Numbers: Parity

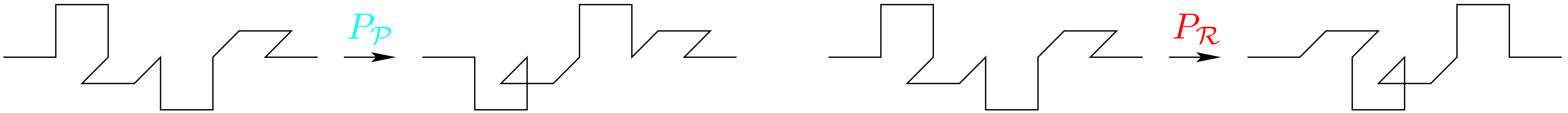
$\mathcal{P}$-Parity reflection plane



$\mathcal{R}$-Parity reflection plane

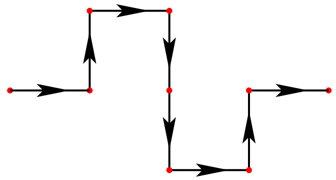


Example:



5. D=3+1: Quantum Numbers: Spin

→ Operator ϕ is given by the trace of path ordered product of blocked links

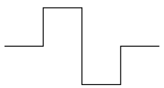


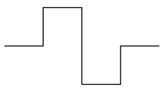
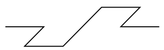
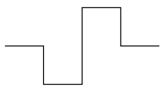
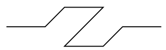
→ We can then form an operator of spin J:

$$\text{Continuum}: \phi(J) = \int d\theta e^{iJ\theta} \phi_\theta \qquad\qquad \text{Lattice}: \phi_L(J) = \sum_n e^{iJn\frac{\pi}{2}} \phi_{n\frac{\pi}{2}}$$

→ Example $J = 1$:

$$\phi_L(J=1) = i\phi_{\frac{\pi}{2}} - \phi_\pi - i\phi_{\frac{3\pi}{2}} + \phi_{2\pi}$$

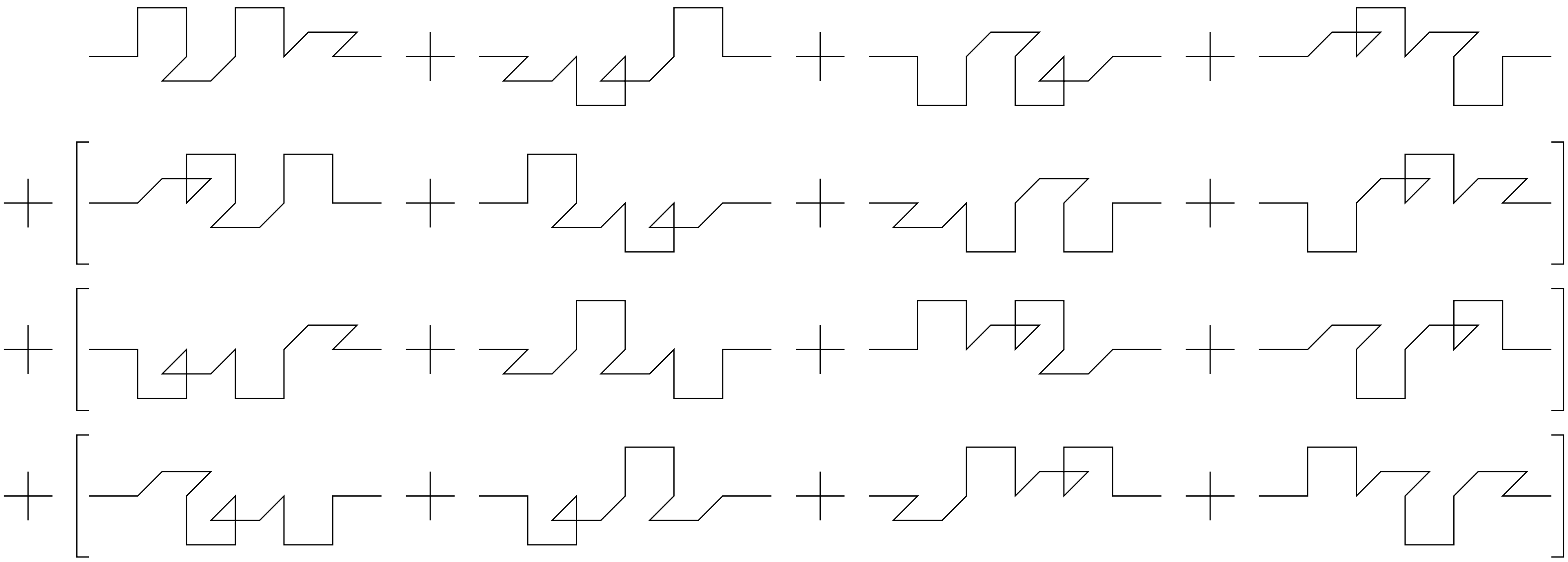
→ If $\phi_{\theta=0} \equiv \mathrm{Tr}\Big\{$  $\Big\}$

$\phi_L(J=1) = \mathrm{Tr}\Big\{$  $+\, i$  $-$  $-\, i$  $\Big\}$

5. D=3+1: Operators: Example $J = 0$, $P_{\mathcal{P}} = +$, $P_{\mathcal{R}} = +$

Irreducible representation: A_1, $P_{\mathcal{R}} = +$.

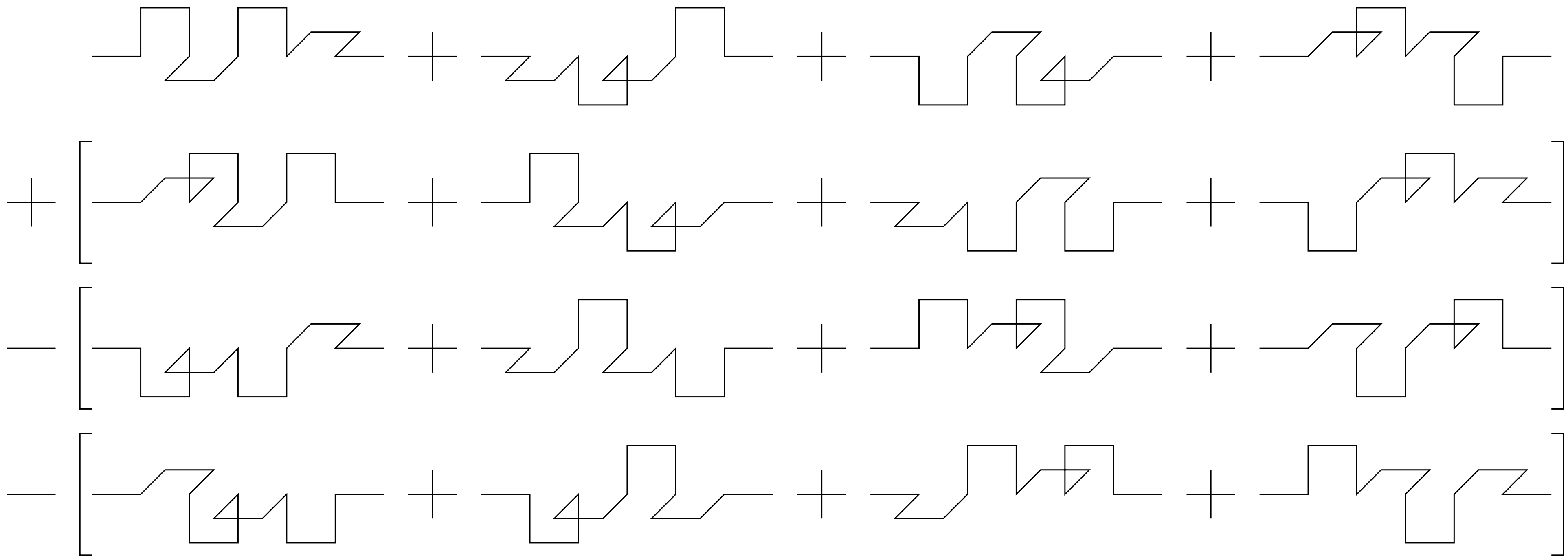
Operator before being traced:



5. D=3+1: Operators: Example $J = 0$, $P_{\mathcal{P}} = -$, $P_{\mathcal{R}} = +$

Irreducible representation: A_2, $P_{\mathcal{R}} = +$.

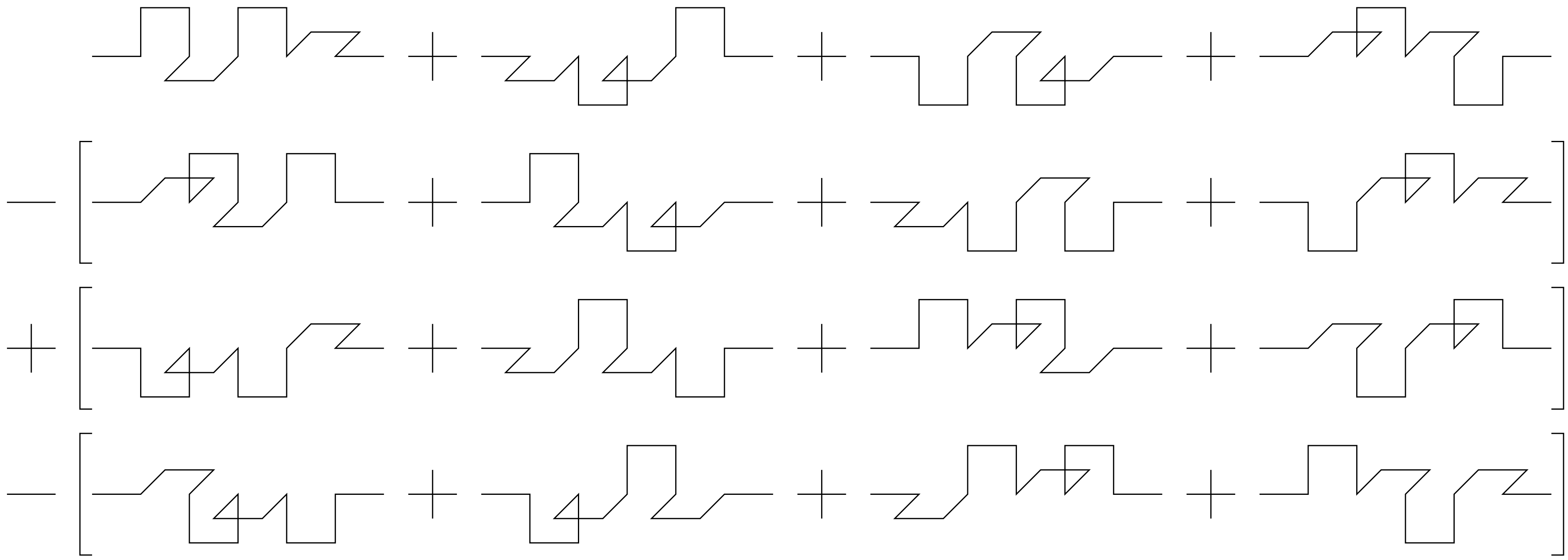
Operator before being traced:



5. D=3+1: Operators: Example $J = 0$, $P_{\mathcal{P}} = +$, $P_{\mathcal{R}} = -$

Irreducible representation: A_1, $P_{\mathcal{R}} = -$.

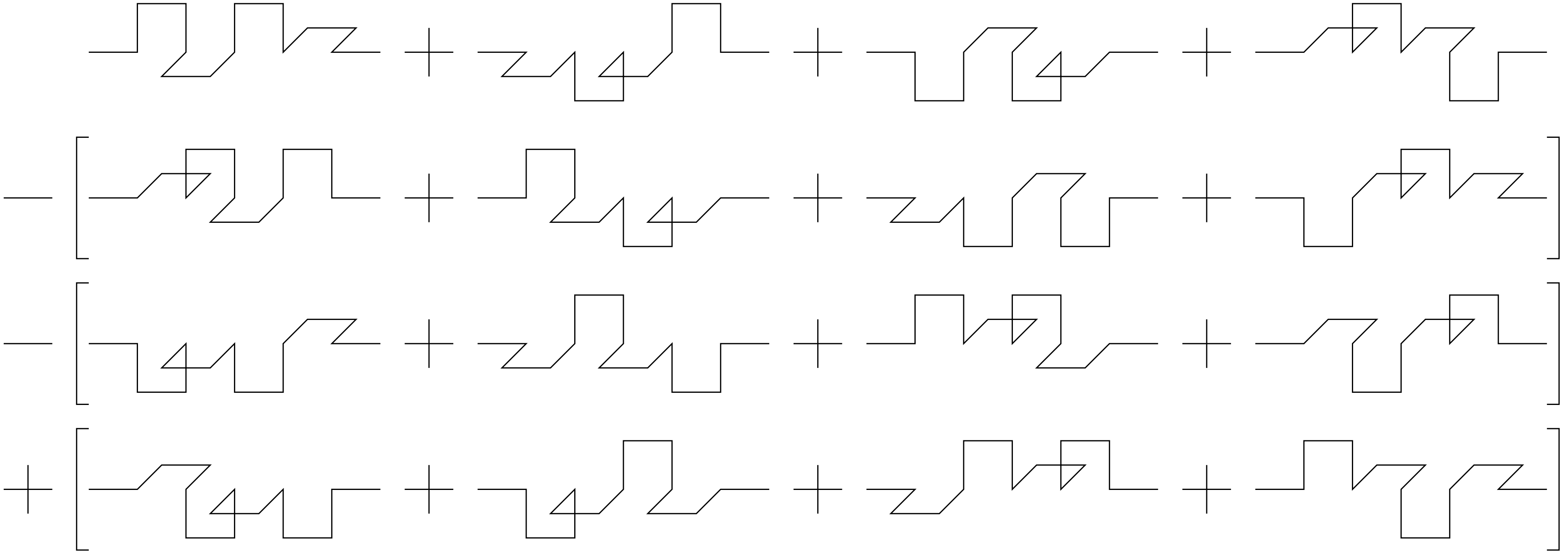
Operator before being traced:



5. D=3+1: Operators: Example $J = 0$, $P_{\mathcal{P}} = -$, $P_{\mathcal{R}} = -$

Irreducible representation: A_2, $P_{\mathcal{R}} = -$.

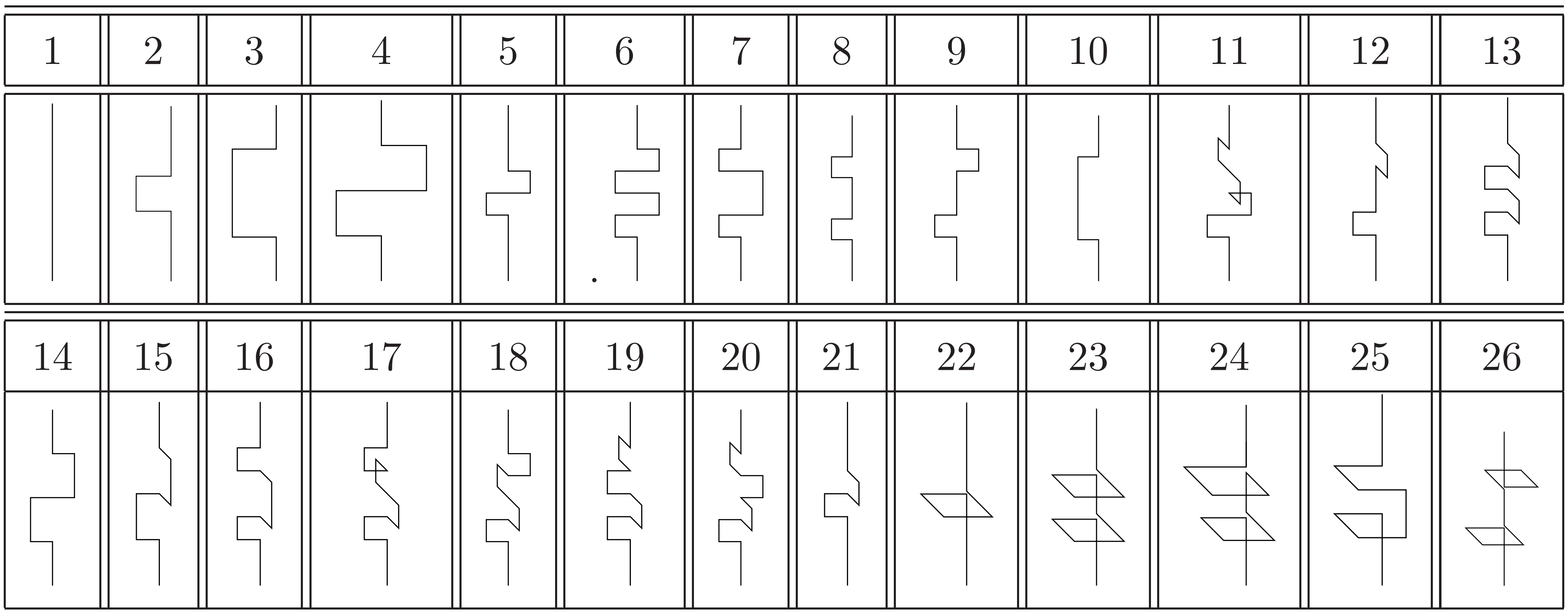
Operator before being traced:



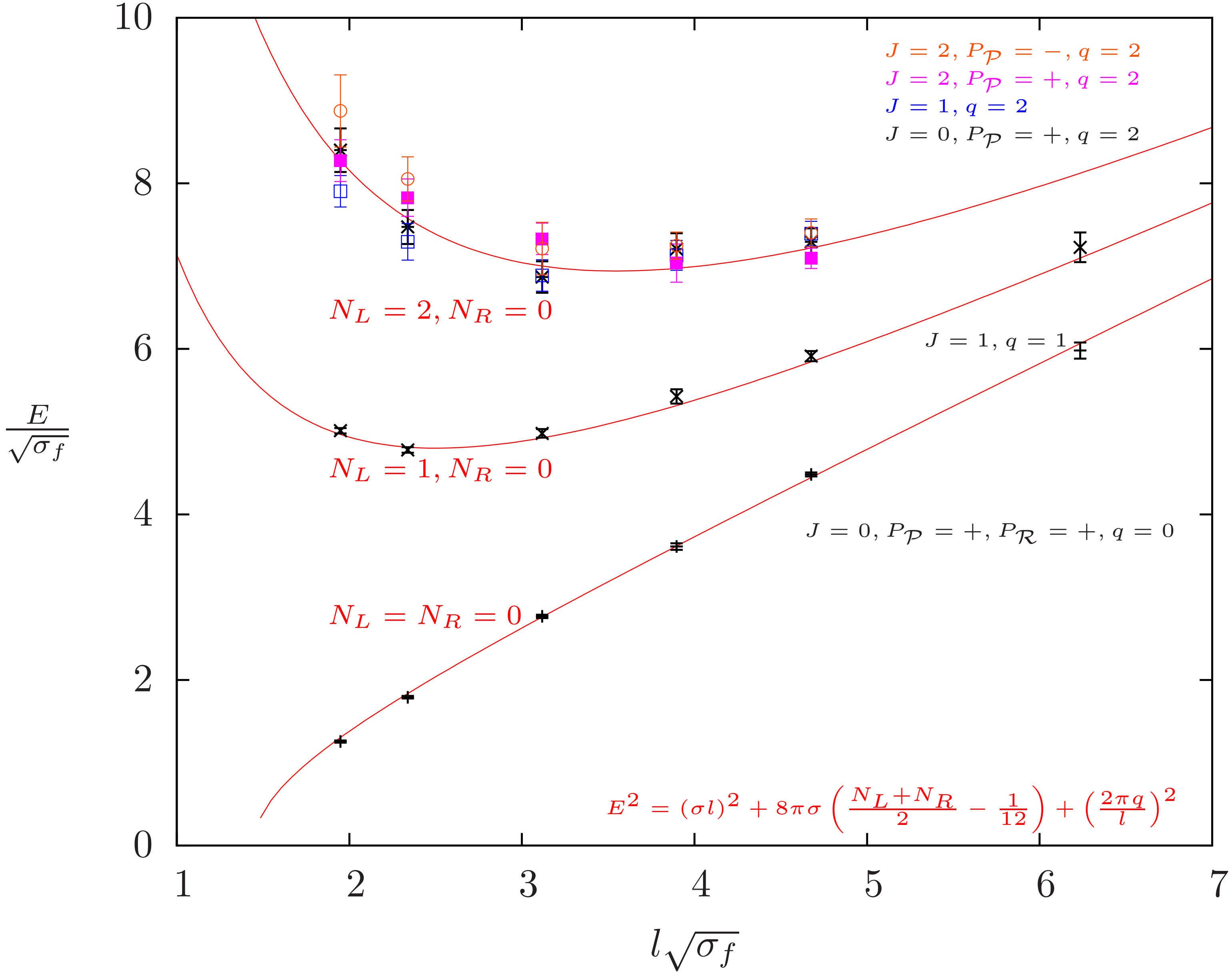
5. D=3+1: Operators: Transverse Deformations

$\longrightarrow \sim 700$ operators

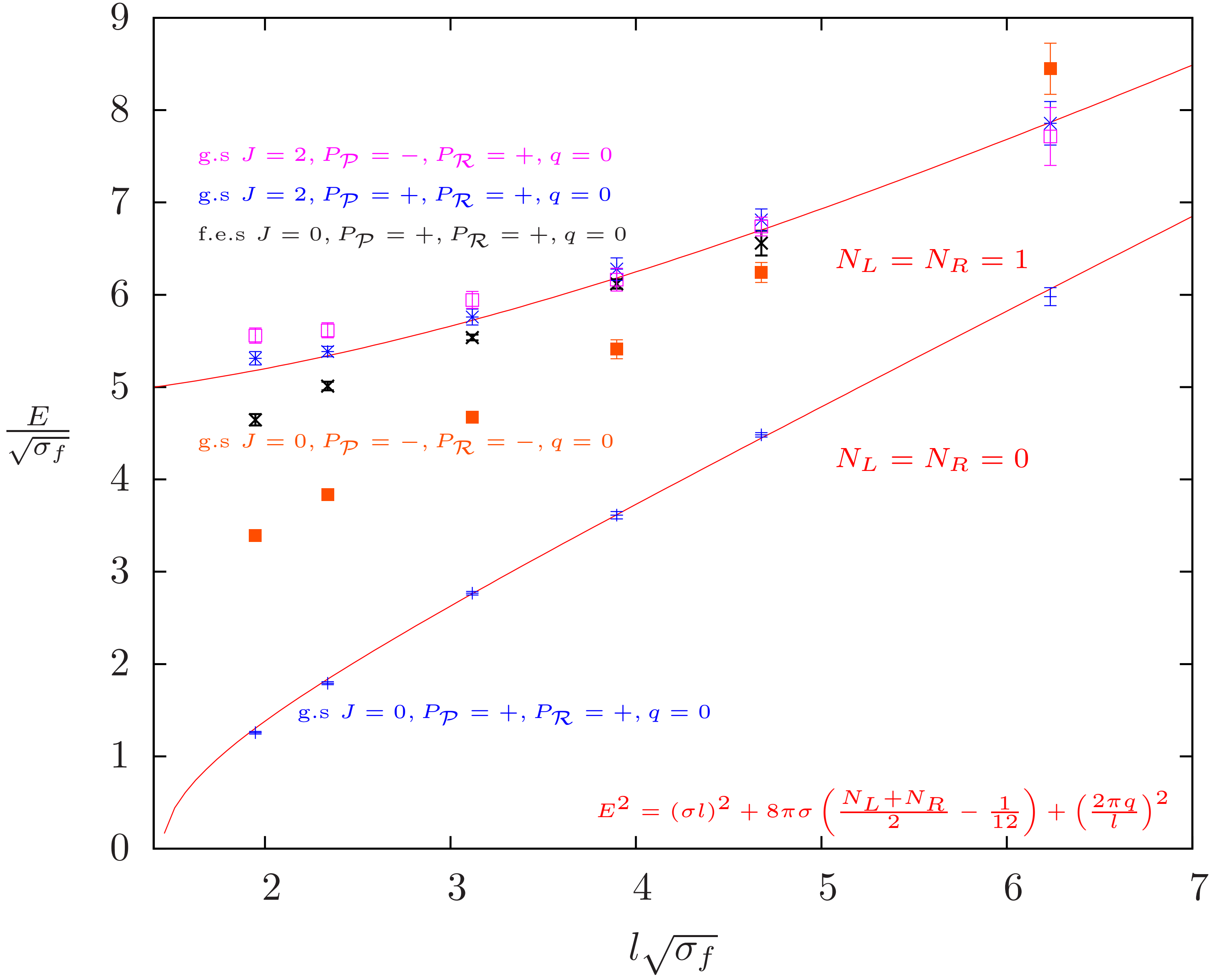
Transverse deformations in 3 spatial dimensions:



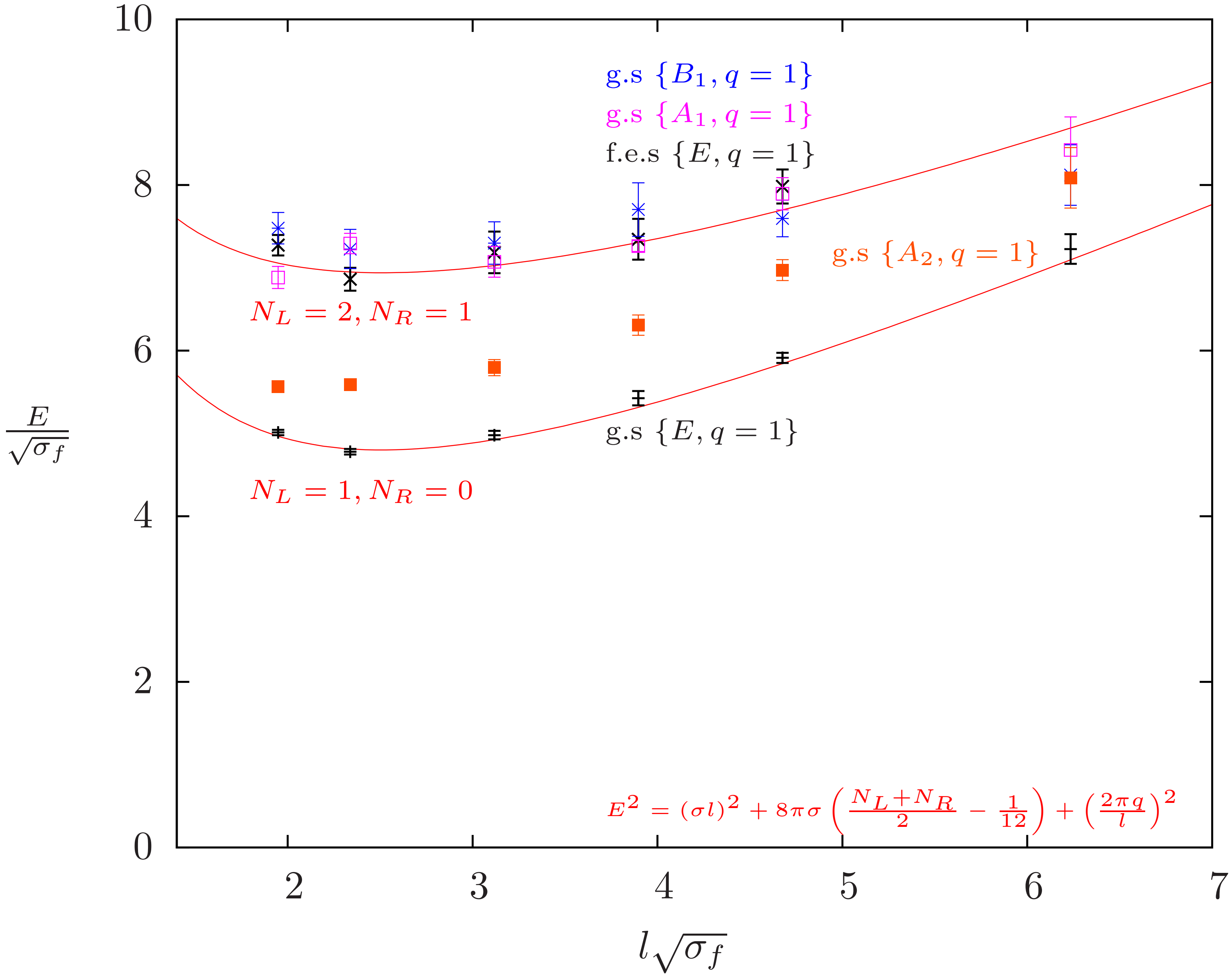
5. D=3+1 Results: $q = 0, 1, 2$ Ground States



5. D=3+1 Results: $q = 0$ Excitations



5. D=3+1 Results: $q = 1$ Excitations



6. Conclusions

- $D = 2 + 1$
  - Closed flux tube can be well described by Nambu-Goto even to small l!
- $D = 3 + 1$
  - The spectrum is mostly closed to Nambu-Goto down to very small l!
  - However, some states (A_2) are far from Nambu-Goto.
  - Non-Stringy dynamics of some kind?