Wilsonian and Holographic Renormalization Groups

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AdS/CFT gives a construction of quantum gravity in AdS spacetime

$$\text{Gravity} \Longrightarrow \text{Gauge Theory} \Longrightarrow \text{Wilson}$$

Outline

- Remarks on the Wilsonian construction of QFT.
- AdS/CFT: Lessons, limitations, and open questions
- Parallels between the Wilson and holographic renormalization groups

Wilsonian construction of QFT

Simple in principle, complicated in practice!

- Separate the path integral variables into high and low energy.
- Integrate out the high energy variables. Result: action with an infinite number of couplings.
- Differentially lower the cutoff scale. One expects that the flow is convergent in all but a finite number of marginal and relevant directions (= renormalizability).
- At zero coupling, this is true by dimensional analysis.
- At nonzero g, renormalizability breaks down only if a dimension shifts by $O(1)$. In perturbation theory this is impossible by definition (note IR, UV cutoffs), QED. It holds to some finite g by continuity.*

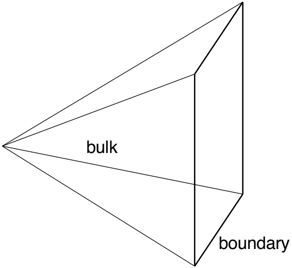
Wilsonian construction of QFT

Simple in principle, complicated in practice!

- Cutoffs do not play nicely with symmetries (gauge, coordinate, extended supersymmetry, dualities!).
- Give up symmetry (Kopper) or work with lattice (Balaban).
- Stability of lattice results is numerical 'proof.'
- Wilson is crude, but currently the only general-purpose tool for defining QFT.
- Is there a better way? E.g. start with 6d CFT?
- NB: there are superrenormalizable examples of gauge/gravity duality (e.g. dimensionally reduced $\mathcal{N} = 4$.

Lessons, limitations, and open questions

- In its current form, the duality reports only the observations of an external observer studying gravity in an anti-de Sitter box:



- In the regime with Einstein gravity, the gauge theory is strongly coupled, so we can't calculate.

Nevertheless, important conceptual lessons have been learned.

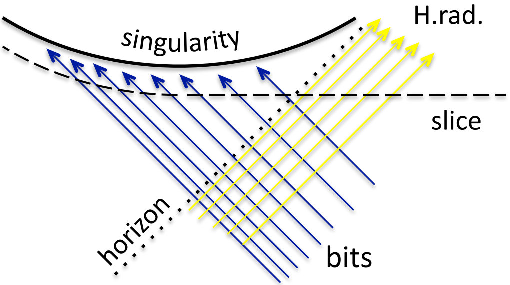
Lessons, limitations, and open questions

One has a construction of gravity in AdS space in which:

- Black hole information is preserved: Hawking radiation is pure.
- The Bekenstein-Hawking entropy counts *all* the states of the black hole.
- Lorentz invariance is preserved.
- Instantonic wormholes do not contribute to the path integral.

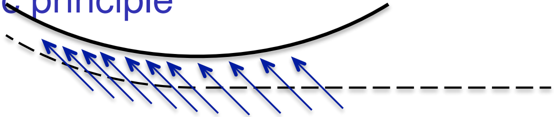
Each of these contradicts assertions that have been made about how quantum gravity must behave. The first two points are closely related, and point to a profound modification of spacetime: gravity is not a Wilsonian theory.

Feeding an evaporating black hole:



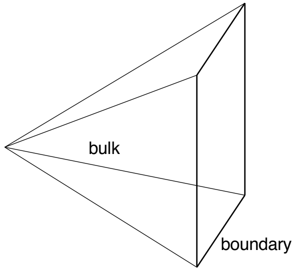
Throw spin-$\frac{1}{2}$ bits into a black hole at a rate that just offsets the evaporation. After N bits, there are at least 2^N possible states, with no upper limit, so eventually this exceeds $e^{S_{\rm BH}}$.

Holographic principle



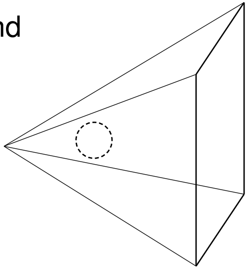
- It is attractive to believe that the BH thermodynamic entropy has a statistical mechanical interpretation, but this requires something radical. Same with the information paradox: black hole complementarity.
- Approaches that reduce to Einstein gravity as an effective field theory on the slice (e.g. closed string field theory, loop quantum gravity, asymptotic safety, dynamical triangulations) start with a Hilbert space that is much too large; they are not *holographic.*
- Holographic principle ('t Hooft, Susskind): quantum gravity in any region should be formulated in terms of a Planckian density of degrees of freedom on its *surface*.

AdS/CFT gives a precise realization of holography, constructing gravity in the bulk in terms of gauge theory on the boundary

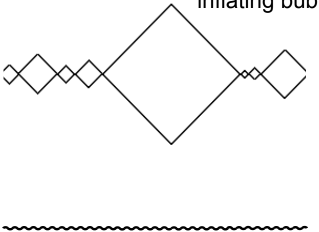


Open questions

We would like to understand holography also for subvolumes in the bulk,



and for cosmological spacetimes



Wilson vs. holography

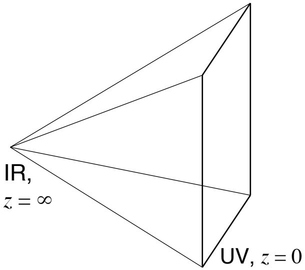
Gravity $\Longrightarrow$ Gauge Theory $\Longrightarrow$ Wilson

Idea: pull the Wilson RG back to the gravity theory, where it becomes the *holographic RG*.

Wilson: integrate out high energy modes progressively.

AdS/CFT maps, E_{CFT} to $1/z$, where z is the emergent coordinate:

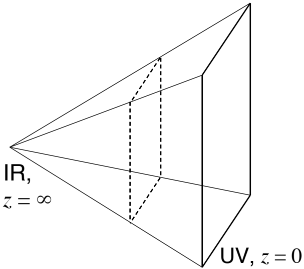
$$ds^2 = L^2 \frac{dz^2 + dx^\mu dx^\mu}{z^2} \,.$$



$z \sim$ size of dipole in BFKL!

Holographic renormalization group

This suggests that we should integrate fields at small z first, and progressively move the cutoff to the IR:



This *holographic RG* idea has been widely studied.

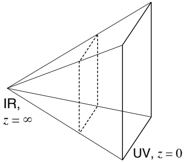
However, much of this departs from the Wilsonian spirit, which we will try to follow (Idse Heemskerk & JP, 1009.xxxx and work in progress).

Outline

1. Splitting the path integral in the bulk.

2. Splitting the path integral in the field theory, and looking for parallels.

1. Splitting the bulk path integral

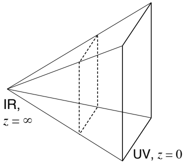


Consider scalar fields $\varphi^I(z,x)$ in a fixed AdS background (later we'll discuss the metric):

$$\begin{aligned} Z &= \int \mathcal{D}\varphi\, e^{-\int_0^\infty dz\, \mathcal{L}(z)} \\ &= \int \mathcal{D}\varphi|_{z>\ell}\, \mathcal{D}\tilde{\varphi}\, \mathcal{D}\varphi|_{z<\ell}\, e^{-\int_\ell^\infty dz\, \mathcal{L}(z) - \int_0^\ell dz\, \mathcal{L}(z)} \\ &= \int \mathcal{D}\tilde{\varphi}\, \Psi_{\mathrm{IR}}(\ell,\tilde{\varphi}) \Psi_{\mathrm{UV}}(\ell,\tilde{\varphi})\,. \end{aligned}$$

where $\tilde{\varphi}^I(x) = \varphi^I(z,x)$. Here ℓ is a length, $\sim 1/$cutoff energy. We want to interpret each factor in this last line.

$\int \mathcal{D}\tilde{\varphi}\, \Psi_{\rm IR}(\ell, \tilde{\varphi}) \Psi_{\rm UV}(\ell, \tilde{\varphi})$



It is plausible to interpret the large-z part of the path integral in terms of a CFT with a UV cutoff (Susskind & Witten, 1998),

$$\Psi_{\rm IR}(\ell, \tilde{\varphi}) = \int \mathcal{D}M|_{k\ell<1} \exp\left\{ - \int d^d x\, \tilde{\varphi}^I(x) \mathcal{O}_I(x) \right\} ,$$

where M stands for the ($N \times N$ matrix) fields of the CFT. This is the usual dictionary between bulk fields and *single-trace* boundary interactions, $\mathcal{O}_I \sim {\rm Tr}(M\partial^2 M \partial M \ldots)$, now with a UV cutoff.

To understand in detail what is nature of the cutoff in the CFT is a hard question. For today, we will just postulate this, and look at the resulting structure.

$\int \mathcal{D}\tilde{\varphi}\, \Psi_{\rm IR}(\ell, \tilde{\varphi}) \Psi_{\rm UV}(\ell, \tilde{\varphi})$

Similarly, it is tempting to identify the UV factor with a Wilsonian action, integrating out the fields above the cutoff scale.

$I_{\rm IR}(\ell, \tilde{\varphi}) = \ln \Psi_{\rm UV}(\ell, \tilde{\varphi})$ is not local, because of propagation of the fields as we integrate in from the boundary, but it is localized on the scale ℓ. Thus it can be expanded in an infinite number of higher derivative local terms (like the Wilsonian action).

$\int \mathcal{D}\tilde{\varphi}\, \Psi_{\rm IR}(\ell, \tilde{\varphi}) \Psi_{\rm UV}(\ell, \tilde{\varphi})$

But what is the role of the functional integral over $\tilde{\varphi}^I$ on the interface? It looks like some weighted average over couplings.

Example: a single scalar supposing that the UV factor is a local gaussian

$$\Psi_{\rm UV}(\ell, \tilde{\varphi}) = \exp\left\{ -\frac{1}{2h} \int d^d x \, (\tilde{\varphi}(x) - g(x))^2 \right\} .$$

Using our postulate for $\Psi_{\rm IR}$ and carrying out the integral over φ gives

$$Z \propto \int \mathcal{D}A|_{k\ell<1} \exp\left\{ - \int d^d x \left(g(x)\mathcal{O}(x) - \frac{h}{2}\mathcal{O}(x)^2 \right) \right\} .$$

I.e., the $\tilde{\phi}$ integral generates *double trace* terms.

$\int \mathcal{D}\tilde{\varphi}\, \Psi_{\rm IR}(\ell,\tilde{\varphi})\Psi_{\rm UV}(\ell,\tilde{\varphi})$

The Wilsonian action is not $\Psi_{\rm UV}(\ell,\varphi)$, but an integral transform,

$$e^{-S_\ell} = \int \mathcal{D}\tilde{\varphi}\, \exp\left\{-\int d^d x\, \tilde{\varphi}^I(x)\mathcal{O}_I(x)\right\}\Psi_{\rm UV}(\ell,\tilde{\varphi})\,.$$

This generates a multi-trace interaction, localized on scale ℓ.

Thus the Wilsonian action necessarily contains multi-trace terms. On the field theory side, pointed out by Li, hep-th/0001193. On the bulk side, by Vielle 1005.4921 (also Faulkner & Liu, unpublished).

RG equations

Varying ℓ gives radial Schrödinger equations

$$\begin{aligned} \partial_\ell \Psi_{\rm IR}(\ell,\tilde\varphi) &= H(\tilde\varphi, \delta/\delta\tilde\varphi)\Psi_{\rm IR}(\ell,\tilde\varphi)\,, \\ \partial_\ell \Psi_{\rm UV}(\ell,\tilde\varphi) &= -H(\tilde\varphi, \delta/\delta\tilde\varphi)\Psi_{\rm UV}(\ell,\tilde\varphi)\,. \end{aligned}$$

The 'Wilsonian action' is the integral transform of $\Psi_{\rm UV}$ and satisfies

$$\partial_\ell e^{-S_\ell} = -H(\delta/\delta\mathcal{O}, \mathcal{O})e^{-S_\ell}\,.$$

This should be compared with the RG on the field theory side.

The path integral is independent of where the splitting is done,

$$0 = \frac{d}{d\ell}Z = \frac{d}{d\ell}\left\langle e^{-S_\ell}\right\rangle_\ell\,.$$

Holographic RG

The 'holographic RG' considered in much of the literature (e.g. de Boer, Verlinde[2] hep-th/9903190) deals only with $\Psi_{\rm IR}$, (which has only single-trace couplings)

$$\partial_\ell \Psi_{\rm IR}(\ell, \tilde{\varphi}) = H(\tilde{\varphi}, \delta/\delta\tilde{\varphi})\Psi_{\rm IR}(\ell, \tilde{\varphi}) \, .$$

But it is not an RG, because of 2nd and higher derivatives in the coupling $\tilde{\varphi}$ (vs. $\beta(g)\partial_g$), from the flow turning on double trace operators.

In the classical approximation, it becomes a first order Hamilton-Jacobi equation for $I_{\rm IR} = \ln \Psi_{\rm IR}$,

$$\partial_\ell I_{\rm IR}(\ell, \tilde{\varphi}) = H(\tilde{\varphi}, \delta I_{\rm IR}/\delta\tilde{\varphi}) \, .$$

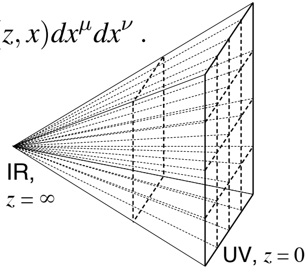
However, this is nonlinear so still not an RG equation. (Also, RG scheme is IR-dependent, not Wilsonian).

Dynamical metric

To treat the metric we can just go to synchronous coordinates,

$$ds^2 = L^2 \frac{dz^2}{z^2} + h_{\mu\nu}(z,x)dx^\mu dx^\nu \ .$$

(Start at AdS boundary). The metric then behaves much like a scalar field.



Ψ_{UV} is not the same as the WDW wavefunction, and does not satisfy the constraints. The WDW wavefunction does not behave like a Wilsonian action because it is nonlocal (the action contains no z derivatives of g_{zz}). Note: problem of observables in GR.

Examples

Common flows studied in AdS/CFT, can be treated in a unified fashion:

Domain wall flow, from one CFT to another, initiated by relevant single-trace perturbation.

Flow from alternate to standard quantization, initiated by relevant double-trace interaction.

2. Splitting the CFT path integral: the Wilson RG

Review:

$$Z = \int \mathcal{D}M\big|_{k<\ell^{-1}} \left(\int \mathcal{D}M\big|_{k>\ell^{-1}} e^{-S} \right) = \int \mathcal{D}M\big|_{k<\ell^{-1}} e^{-S_\ell} \,.$$

The Wilsonian action is localized on the scale ℓ, so expansion in derivatives gives an infinite number of terms.
It satisfies an RG equation,

$$e^{-S_{\ell+d\ell}} = \int \mathcal{D}M\big|_{(\ell+d\ell)^{-1}<k<\ell^{-1}} e^{-S_\ell}$$

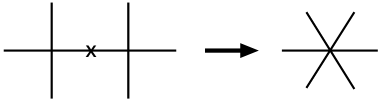
Wilson RG

The general structure of the RG is

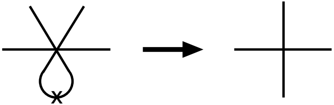
$$\partial_\ell S_I = \int d^d p \, \Delta(p) \left(\frac{\partial S_I}{\partial M(p)} \frac{\partial S_I}{\partial M(-p)} - \frac{\partial^2 S_I}{\partial M(p) \partial M(-p)} \right) ,$$

where S_I is the interaction, Δ is the derivative of the propagator with respect to cutoff. Simple graphical interpretation:

$\dot{S}_I \sim S_I' S_I'$

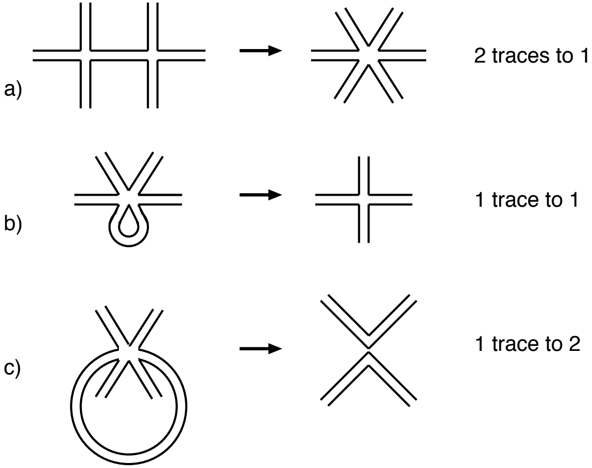


$\dot{S}_I \sim S_I''$



Trace structure

Li hep-th/0001193 noted that Wilson flow generates multi-trace operators, even in the planar limit,



Comparing the RG's

Recall holographic from of RG:

$$\partial_\ell e^{-S_\ell} = -H(\delta/\delta\mathcal{O}, \mathcal{O}) e^{-S_\ell}\ .$$

Wilson RG gives

$$H \sim \mathcal{O}\frac{\delta^2}{\delta\mathcal{O}^2} + \mathcal{O}^2\frac{\delta^2}{\delta\mathcal{O}} + \mathcal{O}\frac{\delta}{\delta\mathcal{O}}\ .$$

This resembles string field theory in radial gauge (cf. Fukuma, Ishibashi, Kawai and Ninomiya, hep-th/9312175).

The supergravity Hamiltonian is more complicated, as we'd expect from integrating out the super-irrelevant operators.

Projection$_1$

In comparing forms of the RG, we should note the effect of projections.

To relate this to the usual RG of renormalizable QFT, separate into

- $\Delta \sim d$: approximately marginal - slow directions of flow
- $\Delta - d = O(1)$: irrelevant - rapidly converging directions of flow

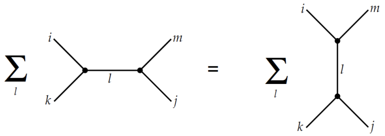
Integrating out the latter gives Callan-Symanzik RG. Compare forms: $\dot{S}_I = S_I' S_I' - S_I''$ vs. all orders.

Projection$_2$

In AdS/CFT one has another separation

- $\Delta \sim d$ or $\Delta - d \leq O(1)$: dual to massless string states
- $\Delta - d = O(\lambda^{1/4})$: dual to stringy states

Aside: in 0907.0151 (Heemskerk, Penedones, JP, Sully) it was conjectured that this large Δ hierarchy plus a large-N expansion were *sufficient* conditions for a CFT to have a gravity dual, and some positive results were obtained.



The Wilson RG would keep all operators. Integrating out only the super-irrelevant operators is dual to the supergravity limit in the bulk.

The cutoff?

We can give a formal answer to the question, what is the cutoff in the gauge theory that maps to the cutoff in z:

$$\begin{aligned}
\Psi_{\rm IR}(\ell,\tilde\phi) &= \int d\tilde\phi'\, G(\ell,\tilde\phi,\tilde\phi')\Psi_{\rm IR}(0,\tilde\phi')\,, \\
\kappa^2\partial_\ell G(\ell,\tilde\phi,\tilde\phi') &= H(\tilde\phi,\tilde\pi)G(\ell,\tilde\phi,\tilde\phi')\,, \\
G(0,\tilde\phi,\tilde\phi') &= \delta(\tilde\phi-\tilde\phi')\,.
\end{aligned} \tag{1}$$

The point is that $\Psi_{\rm IR}(0,\tilde\phi')$ lives at the boundary and can be translated into gauge theory variable by the standard dictionary. In the simplest case of a free scalar in the bulk, the 'cut-off action' is simply the alternate conformal quantization!? It is not clear in what sense this is a cutoff.

Conclusions

- We have identified parallel structures between the Wilson and holographic RG's.
- Notable: multi-traces, synchronous gauge.
- The full Wilson RG is analogous to string theory in radial-temporal gauge.
- The supergravity theory should be compared with a projected RG.
- If we can complete the correspondence we will have a more general formulation of the holographic principle.
- This may also be useful in some of the applications of AdS/CFT to solving strongly coupled field theories.