Checks of strong-weak coupling dualities in N=2 SUSY gauge theories from 2d conformal field theories

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Dramatic recent progress on non-perturbative effects in N=2 d=4 SUSY gauge theories:

- **Generalizations of S-duality**

  `Argyres, Seiberg; Gaiotto; ....`

- **Low-energy particle spectrum ("Wall-crossing")**

  `Gaiotto, Moore, Neitzke; ....`

- **Relation to 2d CFT**

  `Alday, Gaiotto, Tachikawa; ...`

Gaiotto theories I — Lagrangian description

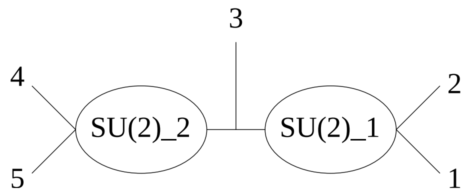
From a large class of N=2 SUSY gauge theories (`Gaiotto`) let us pick an example ($N_c = 2$, $N_f = 5$):

- Gauge group $SU(2) \times SU(2)$

  $\Rightarrow$ Two $SU(2)$ gauge fields A^i_μ + superpartners, esp. scalar fields a^i, $i = 1, 2$.

- Five types of "quarks" Q_r / "anti-quarks" $\tilde{Q}_r$, $r = 1, \ldots, 5$ with their superpartners

To write down the Lagrangian, we need to specify how the quarks couple to the gauge symmetries:

- **Gauge theory** $\mathcal{G}_{2,5}$: 

- **Gauge theory** $\mathcal{G}'_{2,5}$: Same with couplings of quarks Q_2 and Q_3 exchanged.

These theories are UV-finite: Gauge couplings $q_i := e^{-8\pi^2/g_i^2}$ need not to be renormalized.

Gaiotto theories II — S-duality

S-duality conjecture:

Gauge theory $\mathcal{G}_{2,5}$ at strong coupling $q_1 \to \infty$ becomes equivalent to gauge theory $\mathcal{G}'_{2,5}$ provided that the coupling constants are related as

$$q'_1 = \frac{1}{q_1} \qquad q'_2 = q_2 q_1 \,.$$

- Would give control over gauge theory at **strong coupling**.
- Generalizes **Montonen-Olive** duality conjecture.
- Related to generalizations of **electric - magnetic** dualities.
- Is special case of much more general duality conjectures (`Gaiotto`).

Partition functions on S^4

Consider partition function of $\mathcal{G}_{2,5}$ on S^4_R (R: Radius):

$$\mathcal{Z}_{\mathcal{G}_{2,5}}(S^4_R) = \int [\mathcal{D}\Phi] \, e^{-S_{\mathcal{G}_{2,5}}[\Phi]}$$

Only **partially** **protected** by SUSY:

$\mathcal{Z}_{\mathcal{G}_{2,5}}(S^4_R)$ is highly nontrivial function $\mathcal{Z}_{\mathcal{G}_{2,5}}(m; q; R)$ of

- Mass parameters $m = (m_1, \ldots, m_5)$,
- Gauge couplings $q = (q_1, q_2)$, $q_i = e^{-8\pi^2/g_i^2}$, $i = 1, 2$.
- Radius R of S^4.

Partition function on S^4 I — Holomorphic factorization

Based on work of V. Pestun, `AGT` (Alday, Gaiotto, Tachikawa) have shown that partition function of $\mathcal{G}_{2,5}$ on S^4_R can be calculated, and found the form

$$\mathcal{Z}_{\mathcal{G}_{2,5}}(S^4_R) \;=\; \int_{\mathbb{R}} da_1 a_1^2 \int_{\mathbb{R}} da_2 a_2^2 \,\big|\, \mathcal{Z}\big(a;m;R;q\big) \,\big|^2 \,,$$

where $\mathcal{Z}\big(a;m;R;q\big)$, $a=(a_1,a_2)$ captures corrections of the following type:

$$\mathcal{Z} \;=\; \mathcal{Z}_{\text{classical}}\,\mathcal{Z}_{1-\text{loop}}\,\mathcal{Z}_{\text{instanton}}\,,$$

- Tree-level: $\mathcal{Z}_{\text{classical}} = q_1^{a_1^2} q_2^{a_2^2}$,
- Perturbative one-loop corrections $\mathcal{Z}_{1-\text{loop}}$,
- Non-perturbative instanton corrections: $\mathcal{Z}_{\text{instanton}}$

$$\mathcal{Z}_{\text{instanton}} \;=\; \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} q_1^k\, q_2^l\; \mathcal{Z}_{\text{inst}}^{(k,l)}\big(a;m;R\big)\,.$$

Partition function on S^4 II

We know

- **Explicit formulae** for $\mathcal{Z}_{1-\text{loop}}(a; m; R)$

  (from works of `Pestun; Atiyah, Singer ....`).

- **Explicit formulae** for $\mathcal{Z}^{(k,l)}_{\text{inst}}(a; m; R)$

  (from works of `Moore, Nekrasov, Shatashvili; Nekrasov....`).

We do not know

- **how to sum the instanton corrections!**

S-duality

One may similarly calculate

$$\mathcal{Z}_{\mathcal{G}'_{2,5}}(S^4_R) \;=\; \int_{\mathbb{R}} da_1 a_1^2 \int_{\mathbb{R}} da_2 a_2^2 \,\big|\, \mathcal{Z}'\big(a; m; R; q'\big) \,\big|^2 ,$$

where $\mathcal{Z}'\big(a; m; R; q\big) = \mathcal{Z}\big(a; m_5, m_4, m_2, m_3, m_1; R; q\big)$.

To verify **S-duality**, we need to prove that

- $\mathcal{Z}_{\mathcal{G}_{2,5}}(m; R; q)$ can be analytically continued to a neighborhood of $q_1 = \infty$,
- we have

$$\boxed{\mathcal{Z}_{\mathcal{G}_{2,5}}(m; R; q) \;=\; \mathcal{Z}_{\mathcal{G}'_{2,5}}(m; R; q')\,, \quad q_1' = \frac{1}{q_1}\,, \quad q_2' = q_2 q_1\,.}$$

This amounts to a complete **re-summation** of the (non-)perturbative corrections!

Relation to Liouville theory

Main observation (AGT): The result can be written as

$$\boxed{\mathcal{Z}_{\mathcal{G}_{2,5}}(S^4_R) \propto \left\langle e^{2\mu_5\varphi(z_5,\bar{z}_5)} \dots e^{2\mu_1\varphi(z_1,\bar{z}_1)} \right\rangle_1}$$

(up to less interesting factors) where

- $\langle e^{2\mu_5\varphi(z_5,\bar{z}_5)} \dots e^{2\mu_1\varphi(z_1,\bar{z}_1)} \rangle_b$: Correlation function in Liouville theory, formally defined as

$$\left\langle \prod_{r=1}^{5} e^{2\mu_r\varphi(z_r,\bar{z}_r)} \right\rangle_b = \int_{\varphi:\mathbb{P}^1\to\mathbb{R}} [\mathcal{D}\varphi]\, e^{-S_{\mathrm{L}}[\varphi]} \prod_{r=1}^{5} e^{2\mu_r\varphi(z_r,\bar{z}_r)},$$

  where $S[\varphi] = \displaystyle\int_\Sigma \frac{d^2z}{4\pi} \left(\partial_z\varphi\partial_{\bar{z}}\varphi + 4\pi\, M\, e^{2b\varphi}\right)$.

- Parameters related as

$$\mu_r = 1 + i\frac{m_r}{R}\,, \qquad q_1 = \frac{z_{12}z_{35}}{z_{13}z_{25}}\,, \qquad q_2 = \frac{z_{13}z_{45}}{z_{14}z_{35}}\,, \qquad b \equiv 1\,.$$

Liouville theory — Holomorphic factorization

Indeed, correlation fcts. in Liouville theory can also be represented in holomorphically factorized form:

$$\Big\langle \prod_{r=1}^{5} e^{2\mu_r(z_r,\bar{z}_r)} \Big\rangle_b \propto \int_{\mathbb{R}} da_1 \int_{\mathbb{R}} da_2 \, C(\mu;a) \, \big| \, \mathcal{G}\big(a;\mu;b;q\big) \, \big|^2 ,$$

- <u>Conformal block</u> $\mathcal{G}(a;\mu;b;q)$: Power series of the form

$$\mathcal{G}(a;\mu;b;q) \;=\; q_2^{\Delta_2(a;\mu;b)} q_1^{\Delta_1(a;\mu;b)} \left(1 + \sum_{\substack{k,l=0 \\ (k,l)\neq(0,0)}}^{\infty} q_1^k q_2^l \, \mathcal{G}^{(k,l)}(a;\mu;b) \right)$$

  where $\mathcal{G}^{(k,l)}(a;\mu;b)$, Δ_2, Δ_1: **Completely** determined by **conformal symmetry**

- <u>Structure functions</u> $C(\mu;a)$

$$C(\mu;a) \;=\; C(\mu_5,\mu_4,1+ia_2)C(1-ia_2,\mu_3,1+ia_1)C(1-ia_1,\mu_2,\mu_1) \, ,$$

  – <u>**not**</u> determined by **conformal symmetry** alone !

S-duality vs. Locality

Long-standing problem in Liouville theory (BPZ '84):

> Can we choose $C(\mu_3, \mu_2, \mu_1)$ such that **locality** holds,
> $$\left\langle \, \ldots \, e^{2\mu_3(z_3,\bar{z}_3)} e^{2\mu_2(z_2,\bar{z}_2)} \ldots \, \right\rangle_b = \left\langle \, \ldots \, e^{2\mu_2(z_2,\bar{z}_2)} e^{2\mu_3(z_3,\bar{z}_3)} \ldots \, \right\rangle_b \ ?$$

Thanks to the observation made by AGT,

$$\mathcal{Z}_{\mathcal{G}_{2,5}}(S^4_R) \propto \left\langle \, e^{2\mu_5\varphi(z_5,\bar{z}_5)} \ldots e^{2\mu_1\varphi(z_1,\bar{z}_1)} \, \right\rangle_1$$

we find that

> **S-duality** $\Leftrightarrow$ **Locality.**

So if we could prove **locality** in Liouville theory, we'd have a highly nontrivial, **quantitative** check of S-duality in gauge theories.

Liouville theory — Locality

Locality ⇔ highly nontrivial identities of the form

$$\int_{\mathbb{R}} da_1 \int_{\mathbb{R}} da_2 \, C(\mu;a) \, \big| \, \mathcal{G}\big(a;\mu;b;q\big) \, \big|^2 \; = \; \int_{\mathbb{R}} da_1 \int_{\mathbb{R}} da_2 \, C'(\mu;a) \, \big| \, \mathcal{G}'\big(a;\mu;b;q'\big) \, \big|^2 \, ,$$

where $q_1' = \frac{1}{q_1}$, $q_2' = q_2 q_1$, $\mathcal{G}'\big(a;\mu;b;q\big) = \mathcal{G}\big(a;\mu_5,\mu_4,\mu_2,\mu_3,\mu_1;b;q\big)$,

$$C'(\mu;a) \; = \; C(\mu_5,\mu_4,1+ia_2)C(1-ia_2,\mu_2,1+ia_1)C(1-ia_1,\mu_3,\mu_1)\, ,$$

The **locality problem** was finally **solved** `(J.T. '01, '03)` :

> The expression for $C(\mu_3, , \mu_2, \mu_1)$ conjectured by `Dorn, Otto` and `A.B., Al.B. Zamolodchikov` ensures **locality** !

Generalizations – Outlook

These observations can be generalized in various ways:

- More general gauge theories:
  - Any number of $SU(2)$-gauge groups, any number of flavors (done).
  - Higher rank gauge groups (serious problems).
  - Asymptotically free gauge theories ($SU(2)$: in progress).
- Nontrivial observables:
  - Wilson loops, 't Hooft loops.
  - Surface operators.

.... a glimpse on deep connections between:

d=4 SUSY gauge theories

and

integrable models / conformal field theories.