RENORMALIZATION with FLOW EQUATIONS

DESY-WORKSHOP September 2010

1. Introduction
2. Renormalization with Flow Equations
3. A selection of a few rigorous results
4. Renormalization of spontaneously broken gauge theories

joint work with

Georg Keller, Volkhard Müller

Frédéric Meunier, Manfred Salmhofer, Clemens Schophaus, . . .

INTRODUCTION

RENORMALIZATION THEORY

"="

How to obtain well-defined results
for physically significant quantities
to all orders in perturbative
Quantum Field Theory

EXAMPLE:

Make (sufficiently precise) statements on Feynman-amplitudes of gauge theories with massless particles in Minkowski space

Historical remarks

$\geq$ 1929 SELF-MASS of the ELECTRON
Heisenberg-Pauli, Waller-Oppenheimer,
Weisskopf, Weisskopf-Furry

$\geq$ 1947 Bethe, Schwinger, Feynman, Dyson, ...

1960's BPHZ, ...

1970's Zimmermann - Lowenstein, Epstein - Glaser
Faddeev - Popov, Slavnov - Taylor,
't Hooft - Veltman, Lee - Zinn-Justin, BRST, ...

RENORMALIZATION with FLOW EQUATIONS

The Renormalization Group
Wilson $\geq$ 1969

The Flow Equations
Wegner-Houghton 1973

Link Renormalization Theory to the Renormalization Group
Polchinski 1984

The WILSON RENORMALIZATION GROUP

Analyse the theory under a continuous change of scale to pass from the bare (microscopic) to the renormalized (large scale) effective action.

The Wilson effective action is the generating functional of the of the Connected (free propagator) Amputated Schwinger functions (CAS).

RENORMALIZATION of MASSIVE EUCLIDEAN φ_4^4 THEORY

BARE ACTION

$$L_0(\phi) = \int d^4x \, \frac{g}{4!} \, \varphi^4 + \int d^4x \, \{ \frac{\delta m^2}{2} \, \varphi^2 + \frac{\delta Z}{2} \, (\partial_\mu \varphi)^2 + \frac{\delta g}{4!} \, \varphi^4 \}$$

$$\delta m^2 \, , \; \delta g = O(\hbar) \, , \quad \delta Z = O(\hbar^2)$$

REGULARIZED PROPAGATOR

$$C^\Lambda(p) \equiv \frac{1}{p^2 + m^2} \, \{ e^{-\frac{p^2+m^2}{\Lambda_0^2}} - e^{-\frac{p^2+m^2}{\Lambda^2}} \} \, , \quad 0 \leq \Lambda \leq \Lambda_0 \leq \infty$$

$$\dot{C}^\Lambda(p) \equiv \partial_\Lambda C^\Lambda(p) \; = \; - \frac{2}{\Lambda^3} \, e^{-\frac{p^2+m^2}{\Lambda^2}}$$

EFFECTIVE ACTION

$$e^{-\frac{1}{\hbar}L^{\Lambda}(\varphi)} \equiv \mathcal{N} \int d\mu_{\Lambda}(\phi)\, e^{-\frac{1}{\hbar}L_0(\phi+\varphi)}$$

$d\mu_{\Lambda}(\phi)$: Gaussian measure with covariance $\hbar C^{\Lambda}(p)$

FLOW EQUATION

$$\partial_{\Lambda} L^{\Lambda} = \frac{\hbar}{2} \langle \frac{\delta}{\delta\varphi}, \dot{C}^{\Lambda} \frac{\delta}{\delta\varphi} \rangle L^{\Lambda} - \frac{1}{2} \langle \frac{\delta L^{\Lambda}}{\delta\varphi}, \dot{C}^{\Lambda} \frac{\delta L^{\Lambda}}{\delta\varphi} \rangle$$

Expansion in moments of the field φ

$$(2\pi)^{4(n-1)}\ \delta_{\varphi(p_1)} \dots \delta_{\varphi(p_n)} L^{\Lambda}|_{\varphi\equiv 0} = \delta(p_1 + \dots + p_n)\, \mathcal{L}_n^{\Lambda}(p_1, \dots, p_n)$$

Expansion in powers of $\hbar$ (number of loops)

$$\mathcal{L}_n = \sum_{\ell=0}^{\infty} \hbar^{\ell}\, \mathcal{L}_{n,\ell}\ , \qquad \text{note that } \mathcal{L}_{2,0} \equiv 0$$

The FLOW EQUATIONS for the (regularized) CAS

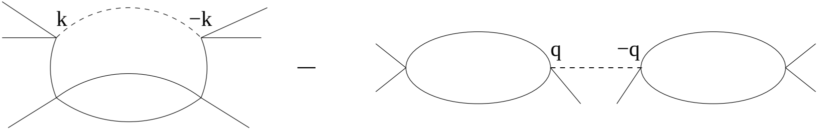
$$\partial_\Lambda \, \mathcal{L}^\Lambda_{n,\ell}(\dots) =$$

$$\frac{1}{2}\int_k \mathcal{L}^\Lambda_{n+2,\ell-1}(k,-k,\dots)\ \dot{C}^\Lambda(k) - \frac{1}{2}\sum_{\ell_i,n_i}\Big[\mathcal{L}^\Lambda_{n_1,\ell_1}\ \dot{C}^\Lambda\ \mathcal{L}^\Lambda_{n_2,\ell_2}\Big]_{sym}(\dots)$$

$$n_1 + n_2 = n+2\,, \quad \ell_1 + \ell_2 = \ell$$

Example :

Contribution to the right hand side of the Flow Equation for $n = 6\,,\ \ell = 2$:



The simplest Boundary Conditions:

For $\underline{\Lambda = \Lambda_0}$ (UV):

$$L^{\Lambda_0} = L_0 \;\Rightarrow\; \mathcal{L}_n^{\Lambda_0} \equiv 0 \quad \text{for} \quad n \geq 5$$

For $\underline{\Lambda = 0}$ (IR):

$$\mathcal{L}_4^0(0) = g\,, \quad \mathcal{L}_2^0(0) = 0\,, \quad \partial_{p^2}\mathcal{L}_2^0(0) = 0$$

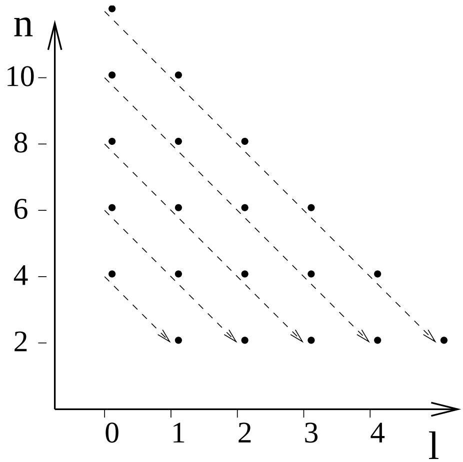
MIXED boundary value problem !

INDUCTIVE SOLUTION

of the renormalization problem :

Inductive scheme :

go up in $n+2\ell$, at fixed $n+2\ell$ go up in ℓ



Induction hypothesis :

$$|\mathcal{L}^{\Lambda}_{n,\ell}(\vec{p})| \leq (\Lambda+m)^{4-n}\ \mathcal{P}_1(\ln\frac{\Lambda+m}{m})\ \mathcal{P}_2(\frac{|\vec{p}|}{\Lambda+m})$$

$$\vec{p} = (p_1, \ldots, p_n)\,,\ \ |\vec{p}| = \sup\{|p_1|, \ldots, |p_n|\}$$

$\mathcal{P}_i$: polynomials with nonnegative coefficients **independent of Λ_0**

Remark :

These bounds are proven *without change* if the bare action is modified by adding irrelevant terms *compatible* with the inductive bounds at $\Lambda = \Lambda_0$.

On the proof :

Irrelevant terms (of mass dimension ≥ 5 are integrated starting at Λ_0 (UV side). Relevant terms (of mass dimension ≤ 4) at the renormalization point are integrated starting from $\Lambda = 0$ (IR side).

Example:

First term on the r.h.s. of the Flow equation for $n \geq 5$:

$$\int_{\Lambda}^{\Lambda_0} d\lambda \int_k \frac{2}{\lambda^3}\, e^{-\frac{k^2+m^2}{\lambda^2}}\, (\lambda+m)^{4-(n+2)} \mathcal{P}_1(\ln \frac{\lambda+m}{m})\, \mathcal{P}_2(\frac{|(\vec{p},k,-k)|}{\lambda+m})$$

$$\leq \int_{\Lambda}^{\Lambda_0} d\lambda\, (\lambda+m)^{4+1-(n+2)}\, \mathcal{P}_1(\ln \frac{\lambda+m}{m})\, \tilde{\mathcal{P}}_2(\frac{|\vec{p}|}{\lambda+m})$$

$$\leq (\Lambda+m)^{4-n}\, \tilde{\mathcal{P}}_1(\ln \frac{\Lambda+m}{m})\, \tilde{\mathcal{P}}_2(\frac{|\vec{p}|}{\Lambda+m})$$

A SELECTION of a FEW RIGOROUS RESULTS

IMPROVING the BOUNDS :

For K sufficiently large we have for $\Lambda = 0$

$$|\mathcal{L}^{0}_{n,\ell}(\vec{p})| \;\leq\; K^{2\ell+n}\,(n+\ell)!\,\sum_{\lambda=0}^{\lambda=\ell} \frac{\log^{\lambda}(\frac{|\vec{p}|+m}{m})}{\lambda!}\;, \quad n \geq 4$$

Remark :

For a viable induction hypothesis we have to specify the dependence on Λ .

RENORMALIZATION in MINKOWSKI SPACE

$$\frac{1}{p_{eu}^2 + m^2} \rightarrow \frac{1}{p_{rel}^2 - m^2 + i\varepsilon(p_{eu}^2 + m^2)}$$

- through analytic continuation $\surd$

Theorem :

The four-point function in Minkowski space

$$\mathcal{L}_4(p_1, \ldots, p_4)$$

is (order by order in $\hbar$) a continuous function of its independent external momenta.

RENORMALIZATION on RIEMANNIAN MANIFOLDS $\surd$

Makes use of efficient bounds on the heat kernel on Riemannian manifolds
(geodesically complete, simply connected manifolds with sectional curvatures bounded above and below, *Euclidean signature !*)

Presently there seems to be *no way to carry over* the proof *to* pseudo-Riemannian manifolds with *Lorentz signature* since generally there is no analytic continuation from the Euclidean signature case.

The method can also handle with

THEORIES with MASSLESS PARTICLES.

In this case it is necessary to write inductive bounds controlling inductivley the *singularities* of the Schwinger functions *at* arbitrarily *exceptional external momenta.*

- *work in progress with Riccardo Guida* -

So far there seems to be *no rigorous Flow equation based proof of the renormalizability of QCD.*

RENORMALIZATION OF SPONTANEOUSLY BROKEN GAUGE THEORIES

The regularization in momentum space breaks gauge invariance

$\Rightarrow$

The Flow is not gauge invariant

STRATEGY :

Enlarge the space of theories abandoning local gauge invariance. Show that in the enlarged space of theories there exist boundary conditions for the flow - or equivalently that there exists a bare action - such that the renormalized theory is gauge invariant, i.e. that the correlation functions satisfy the Slavnov-Taylor-identities in the limit $\Lambda \to 0\,,\ \Lambda_0 \to \infty$.

THE SU(2)-YANG-MILLS-HIGGS-MODEL

$$S_{\rm inv} = \int \left\{ \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} (\nabla_\mu \phi)^* \nabla_\mu \phi + \lambda (\phi^* \phi - \rho^2)^2 \right\}$$

Yang-Mills field $\{A^a_\mu\}_{a=1,2,3}$

Complex scalar doublet $\{\phi_\alpha\}_{\alpha=1,2}$ written as

$$\phi(x) = \begin{pmatrix} B^2(x) + iB^1(x) \\ \rho + h(x) - iB^3(x) \end{pmatrix}$$

Higgs mechanism produces masses

$$m = \frac{1}{2} g \rho \,, \quad M = (8\lambda\rho^2)^{\frac{1}{2}}$$

't Hooft gauge fixing

$$S_{\mathrm{g.f.}} = \int \frac{1}{2\alpha}(\partial_\mu A^a_\mu - \alpha m B^a)^2$$

Faddeev-Popov term

$$S_{\mathrm{gh}} =$$

$$-\int \bar{c}^a \{(-\partial_\mu\partial_\mu + \alpha m^2)\delta^{ab} + \frac{\alpha}{2} gm\, h\delta^{ab} + \frac{\alpha}{2} gm\epsilon^{acb} B^c - g\partial_\mu \epsilon^{acb} A^c_\mu\} c^b$$

TOTAL "CLASSICAL ACTION"

$$S_{\mathrm{BRS}} = S_{\mathrm{inv}} + S_{\mathrm{g.f.}} + S_{\mathrm{gh}}$$

PROPAGATORS

$$C^{ab}_{\mu\nu}(p) \;=\; \frac{\delta^{ab}}{p^2+m^2}\{\delta_{\mu\nu}-(1-\alpha)\frac{p_\mu p_\nu}{p^2+\alpha m^2}\}$$

$$C(p) \;=\; \frac{1}{p^2+M^2}\,, \qquad S^{ab}(p) \;=\; \frac{\delta^{ab}}{p^2+\alpha m^2}$$

REGULARIZATION

$$\sigma_\Lambda(p^2) := \exp\{-\frac{1}{\Lambda^{10}}[(p^2+m^2)(p^2+\alpha m^2)(p^2+M^2)(p^2)^2]\}$$

$$\sigma_{\Lambda,\Lambda_0}(p^2) := \sigma_{\Lambda_0}(p^2)-\sigma_\Lambda(p^2)$$

Regularized propagators

$$C^\Lambda(p^2) := C(p^2)\;\sigma_{\Lambda,\Lambda_0}(p^2)$$

Following our strategy we consider *the most general bare interaction* L_0 with the given field content and respecting all the global symmetries of the theory. This bare interaction L_0 contains a priori 37 relevant parameters of dimension ≤ 4 . The Slavnov-Taylor identities are then violated due to the presence of the momentum space regularization and due to the BRS-noninvariance of the bare interaction.

From the regularized path integral- choosing $\Lambda = 0$ and $\Lambda_0 < \infty$ - we then deduce the following

VIOLATED SLAVNOV-TAYLOR-IDENTITIES (VSTI)

$$\langle \frac{\delta \Gamma}{\delta \underline{A}^a_\mu}, \sigma_{0,\Lambda_0} \frac{\delta \Gamma}{\delta \gamma^a_\mu} \rangle + \langle \frac{\delta \Gamma}{\delta \underline{h}}, \sigma_{0,\Lambda_0} \frac{\delta \Gamma}{\delta \gamma} \rangle + \langle \frac{\delta \Gamma}{\delta \underline{B}^a}, \sigma_{0,\Lambda_0} \frac{\delta \Gamma}{\delta \gamma^a} \rangle + \langle \frac{\delta \Gamma}{\delta \underline{c}^a}, \sigma_{0,\Lambda_0} \frac{\delta \Gamma}{\delta \omega^a} \rangle$$

$$- \langle (\frac{1}{\alpha} \partial_\nu \underline{A}^a_\nu \; - \; m \, \underline{B}^a), \sigma_{0,\Lambda_0} \frac{\delta \Gamma}{\delta \underline{\bar{c}}^a} \rangle = \mathbf{\Gamma}_1(\underline{A}, \underline{h}, \underline{B}, \underline{c}, \underline{\bar{c}})$$

(γ, ω are sources for the BRS-insertions)

Here $\mathbf{\Gamma}_1(\underline{A}, \underline{h}, \underline{B}, \underline{c}, \underline{\bar{c}})$ - more precisely $\mathbf{\Gamma}_1{}^{0,\Lambda_0}(\underline{A}, \underline{h}, \underline{B}, \underline{c}, \underline{\bar{c}})$ - is the value at $\Lambda = 0$ of the generating functional of 1PI correlation functions containing one composite operator insertion. At the upper boundary value $\Lambda = \Lambda_0$ this composite operator insertion is obtained through BRS-variation of the noninvariant bare interaction and of the regularized quadratic (free) part of the action

$$L_1^0\, \varepsilon = -\delta_{BRS}\, (Q^{0,\Lambda_0} + L^0)$$

It thus respects the global symmetries of the theory, has ghost number one, and the operator insertion is of dimension 5. There exist a priori 53 relevant contributions to this type of insertion with dimension ≤ 5.

Our **Task** is to make vanish $\mathbf{\Gamma}_1{}^{0,\Lambda_0}(\underline{A}, \underline{h}, \underline{B}, \underline{c}, \underline{\bar{c}})$ for $\Lambda \to 0$ and $\Lambda_0 \to \infty$.

To achieve the task we use the fact that the inserted functional also obeys a Flow equation obtained similarly as the noninserted one. We then use the following evanescence statement

Proposition : Let ${\Gamma_1}_{n,l}^{\Lambda,\Lambda_0}$ be the 1PI Schwinger functions corresponding to an operator insertion of dimension D. Suppose that the following restrictions on the boundary conditions are verified :

$$(\Gamma_1)_{n,l}^{\Lambda_0,\Lambda_0} \leq \Lambda_0^{D-|n|-|w|}\,\mathcal{P}_1(log\frac{\Lambda_0}{m})\,\mathcal{P}_2(\frac{|\vec{p}|}{\Lambda_0}) \;\; \text{for} \;\; n > D \;,$$

$$(\Gamma_1)_{n,l}^{0,\Lambda_0}(\vec{p}=0) = 0 \;\; \text{for} \;\; n \leq D \;.$$

Then the 1PI Schwinger functions obey the following bounds

$$(\Gamma_1)_{n,l}^{\Lambda,\Lambda_0} \leq \frac{\Lambda+m}{\Lambda_0}\;(\Lambda+m)^{D-|n|-|w|}\,\mathcal{P}_1(log\frac{\Lambda_0}{m})\,\mathcal{P}_2(\frac{|\vec{p}|}{\Lambda+m})$$

This bound implies that the 1PI Schwinger functions vanish in the limit $\Lambda_0 \to \infty$, in particular at the physical point $\Lambda = 0$.

As regards the VSTI we thus can limit ourselves to make vanish the 53 relevant contributions to $\mathbf{\Gamma_1}^{\Lambda,\Lambda_0}(\underline{A}, \underline{h}, \underline{B}, \underline{c}, \underline{\bar{c}})$. To do so we dispose of the 37 renormalization conditions to be imposed on the functional Γ^{0,Λ_0}. Besides there are (up to) 7 normalization constants for the bare BRS- transformations we may dispose of.

So the Task now is :

Make vanish 53 constants using 37+7 constants, maintaining finiteness and leaving arbitrary the free parameters of the theory, i.e.

$$g, \ \lambda, \ m,$$

the gauge fixing parameter α, and the normalizations of the fields.

Result :

The system of equations for the relevant part of the VSTI admits the following solution :

The relevant part of $\Gamma_1{}^{0,\Lambda_0}(\underline{A}, \underline{h}, \underline{B}, \underline{c}, \underline{\bar{c}})$ *can be made vanish while*

1) choosing freely the couplings $g, \lambda,$ *and the parameters* ρ, α

2) choosing freely the wave function renormalizations

3) choosing freely the normalization of one of the BRS-transformations

All other renormalization constants are uniquely fixed by requiring the boundary conditions of the evanescence theorem.