Quantum fields on curved spacetimes

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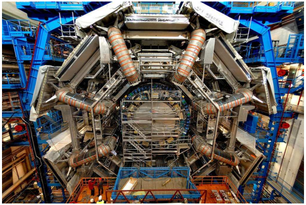
Outline

What is QFT good for?

1. QFT was in invented to describe Elementary Particles and their collisions in particle accelerators.
2. Some of its predictions are **extremely** precise!
3. Not fully understood in non-perturbative regime. One of the Maths "Clay-Problems" worth USD 1.000.000

www.claymath.org/millennium/



Famous "$(g-2)$"-parameter [Kinoshita et al., Dehmelt et al.]:

$$a_{e^-} = 1159652175.86(8.84) \times 10^{-12}$$

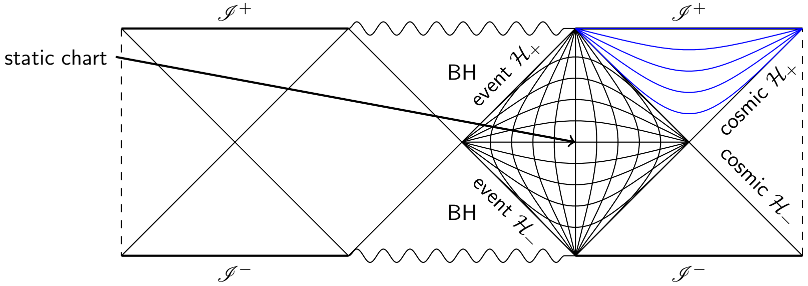
$$\Delta a_{e^-} \leq 12.4(9.5) \times 10^{-12}$$

The significance of QFT goes beyond this!

1. Close relationship to statistical mechanics models
2. Dual description of quantum gravity in AdS-CFT correspondence
3. Strings are described by $2D$-QFT's
4. Developments in mathematics (Functional Analysis, Topology, ...)
5. Black-hole radiance or *macroscopic* effects in early Universe cosmology ("primordial fluctuations") → QFT on manifolds. **This talk!**

WHAT ARE TYPICAL QUESTIONS IN QFT ON CST?

There are many questions about the behavior of QFT's on particular spacetimes, e.g. Schwarzschild-deSitter:



$$ds^2 = -\left(1-2Mr^{-1}-\Lambda r^2\right)d\eta^2+\left(1-2Mr^{-1}-\Lambda r^2\right)^{-1}dr^2+r^2d\omega_2^2\ .$$

QUESTIONS

What is the behavior of correlators $\langle\phi(t,x_1)\ldots\phi(t,x_n)\rangle_\Psi$ in the blue region for large times (DeSitter: → power-spectrum)?

What are typical questions in QFT on CST?

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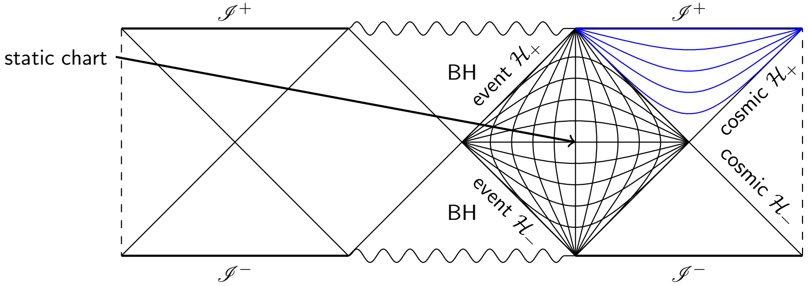

$$ds^2 = -\left(1 - 2Mr^{-1} - \Lambda r^2\right) d\eta^2 + \left(1 - 2Mr^{-1} - \Lambda r^2\right)^{-1} dr^2 + r^2 d\omega_2^2 \, .$$

Questions

What is the behavior of correlators in the static chart for $x_i \to \infty$ ($\to$ BH-radiance)?

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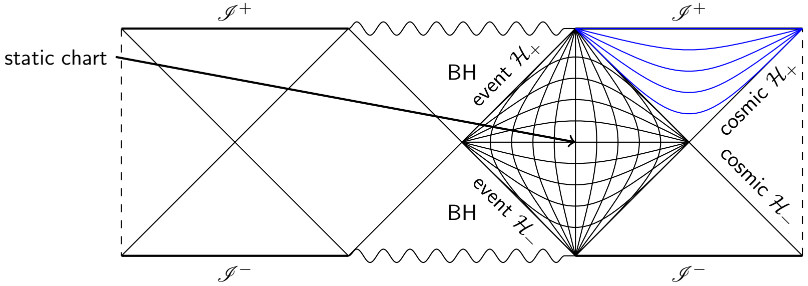

$$ds^2 = -\left(1 - 2Mr^{-1} - \Lambda r^2\right) d\eta^2 + \left(1 - 2Mr^{-1} - \Lambda r^2\right)^{-1} dr^2 + r^2 d\omega_2^2 \ .$$

Questions

What is the behavior of e.g. $\langle T_{\mu\nu} \rangle_\Psi$ near singularity?

What are typical questions in QFT on CST?

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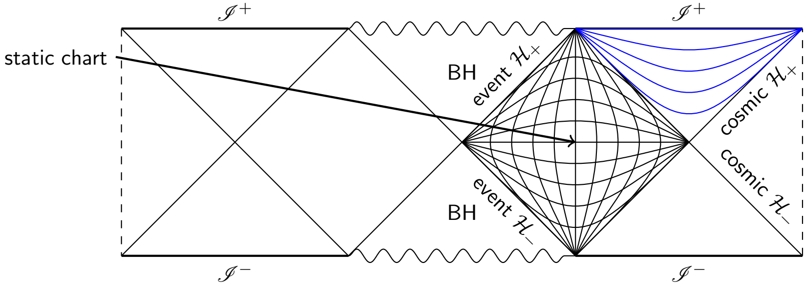

$$ds^2 = -\left(1-2Mr^{-1}-\Lambda r^2\right)d\eta^2+\left(1-2Mr^{-1}-\Lambda r^2\right)^{-1}dr^2+r^2d\omega_2^2\ .$$

Questions

What is the behavior of the "flux" $f_\mu = -\langle T_{\mu\nu}\dot{\gamma}^\nu\rangle_\Psi$ near $\mathcal{H}$?

What are typical questions in QFT on CST?

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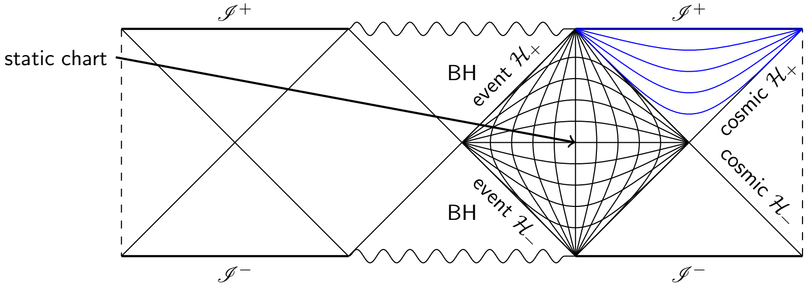

$$ds^2 = -\left(1-2Mr^{-1}-\Lambda r^2\right)d\eta^2+\left(1-2Mr^{-1}-\Lambda r^2\right)^{-1}dr^2+r^2d\omega_2^2\ .$$

Questions

What state(s) Ψ should be we consider?

What are typical questions in QFT on CST?

There are important general questions about the behavior of QFT's on general spacetimes, such as:

General questions

1. Is there a vacuum state? (no, in general)
2. Is there a notion of particle? (no, in general)
3. Does the spin-statistics relation/PCT hold? (yes)
4. Can one renormalize in a consistent way? (always yes)
5. Is there a Euclidean formulation? (no, in general)
6. What is the meaning of "general covariance"?
7. What is the physical interpretation of the 'theory'?
8. What is the meaning of positivity of energy?
9. How to incorporate backreaction effects?

General features of correlators

As in flat Minkowski spacetime, we could say that a QFT on a curved spacetime (M, g) is specified by a collection of correlators

$$\langle \mathscr{O}_{i_1}(x_1) \dots \mathscr{O}_{i_n}(x_n) \rangle_\Psi$$

where:

1. There is one set of correlators for each state Ψ.
2. The $\mathscr{O}_i$'s are composite fields, e.g. $\phi, T_{\mu\nu}, J_\mu, \dots$.
3. We expect there to be singularities if x_i is on x_j's lightcone.
4. We expect that fields should (anti-) commute if x_i and x_{i+1} are spacelike.
5. We want the state to be "positive" ("unitarity"): $\langle A^* A \rangle_\Psi \geq 0$ for any expression such as (f a smearing function)

$$A = \int f(x_1, \dots, x_n) \prod \phi(x_i) \ .$$

General features of correlators

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But:

1. There are many examples of pathological states satisfying these criteria (e.g. α-states in deSitter, "instantaneous ground state" in FRW-universe,...).
2. How to recognize that correlators belong to *different state* but *same theory*? (Operator-product-expansion, OPE)
3. Does not incorporate any notion of positive energy (microlocal spectrum condition).
4. How to formulate notion of "general covariance" (OPE)?

All states should satisfy the operator product expansion:

[Wilson,Kadanoff], [Zimmermann], [Keller et al.],[Belavin et al.],...,[curved space: S.H.]

$$\langle \mathscr{O}_{j_1}(x_1)\cdots\mathscr{O}_{j_n}(x_n)\rangle_\Psi \sim \sum \underbrace{C^i_{j_1\dots j_n}(x_1,\dots,x_n;y)}_{\text{state indep.}}\underbrace{\langle\mathscr{O}_i(y)\rangle_\Psi}_{\text{state dep.}}$$

- **Physical idea:** Behavior of the theory at short distances "factorizes" from the behavior at long distances/from properties of specific state.
- **Covariance of theory:** Coefficients are local functionals of the metric, curvature etc.

Example

Free KG-field

$$\phi(x)\phi(y) \sim \left\{\frac{u}{\sigma + i0t} + \sum_n v_n \sigma^n \log(\sigma + i0t)\right\}\mathbf{1} + \phi^2(y) + \dots$$

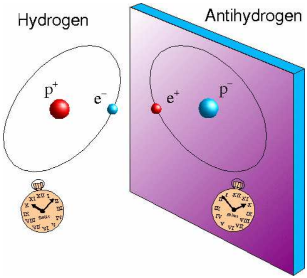
where dots represent regular terms, and where u, v_n are the *Hadamard deWitt coefficients*.

Fundamental theorems

Fundamental theorems about QFT on curved spacetimes e.g. **P**arity-**T**ime-**C**harge [S.H. 2004, & Wald 2008]:

But: What actually is "P" and "T" in a spacetime without reflection symmetries, such as e.g. expanding FRW:

$$ds^2 = -d\tau^2 + a(\tau)^2(dx_1^2 + dx_2^2 + dx_3^2) \quad ?$$



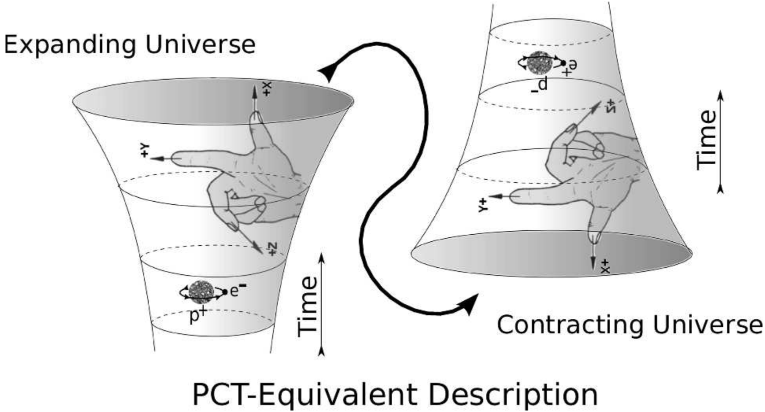
Minkowski spacetime:

$U|\text{in, momenta}\rangle = |\text{out}, -\text{momenta}\rangle^C$.

Curved spacetime: **No "same Universe" formulation of PCT:** Theorem connects "in"-state in one universe with "out state" in universe with opposite space and time orientations!

PCT=symmetry of OPE!!!

The PCT-theorem in curved space



The PCT theorem connects the QFT in one spacetime (e.g. expanding universe) to that of the spacetime with opposite time- and space- orientation (contracting universe).

The PCT-theorem in curved space

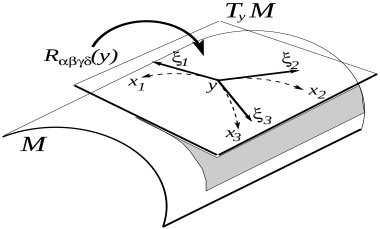
As a **technical statement** about the OPE-coefficients, the PCT-theorem can be stated as saying that

$$C^{j}_{i_1 \dots i_n}[T, o] = \overline{C^{\bar{j}}_{\bar{i}_n \dots \bar{i}_1}[-T, -o]}$$

where o, T are the orientation (o) and time orientation (T) of the underlying spacetime (M, g).

The **proof** of the PCT-theorem relies on fundamental properties of the OPE:

- Curvature expansion: (e.g. $u/\sigma \sim 1/\xi^2 + R_{\mu\nu}\xi^\mu \xi^\nu/\xi^2 + \dots$)



THE PCT-THEOREM IN CURVED SPACE

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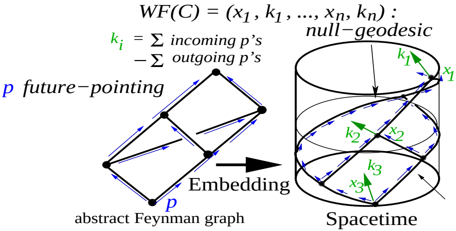
The **proof** of the PCT-theorem relies on fundamental properties of the OPE:

- μ-local spectrum condition: [Brunetti et al. 1998,2000, S.H. 2006]



The PCT-theorem in curved space

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where o, T are the orientation (o) and time orientation (T) of the underlying spacetime (M,g).
The **proof** of the PCT-theorem relies on fundamental properties of the OPE:

- Associativity of OPE ("crossing symmetry")

DeSitter spacetime

DeSitter space can be defined as the 4-dimensional submanifold

$$dS_4 = \{-X_0^2 + X_1^2 + X_2^2 + X_3^2 + X_4^2 = H^{-2}\}$$

of $5D$-Minkowski space.

DeSitter space is

- A space of constant curvature H^2, conformally flat.
- Has isometry group $O(4,1)$.
- The signed squared geodesic distance σ between two points $X, Y \in dS_4$ is given by

$$\cos(H\sqrt{\sigma}) = H^2\, X \cdot Y \equiv Z.$$

The quantity Z is called the "point-pair invariant" and can be used nicely to illustrate the causal relationships in dS_4:

DeSitter spacetime

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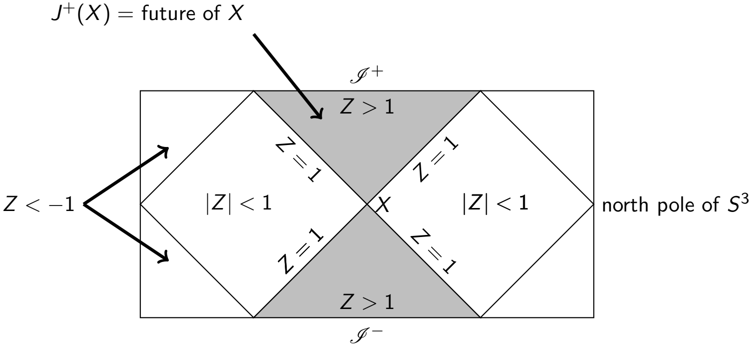
of 5D-Minkowski space.



DeSitter spacetime

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of $5D$-Minkowski space.

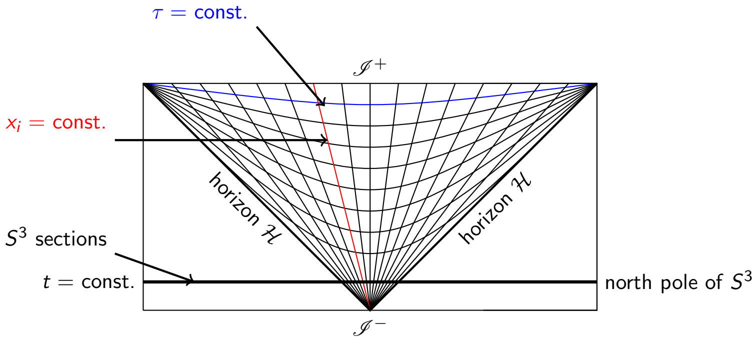
The drawing represents a *conformal diagram* of deSitter, where the line-element has been brought to the form

$$\begin{aligned} ds^2 &= -dt^2 + H^{-2}\cosh^2 Ht \; d\omega_3^2 \\ &= H^{-2}\Omega^{-2}\left(-d\Omega^2 + d\omega_3^2\right) \end{aligned}$$

The COSMIC NO-HAIR CONJECTURE states that, away from black holes, essentially *any* solution to $G_{\mu\nu} = \Lambda g_{\mu\nu}$ will locally approach deSitter space with $\Lambda = 3H^2$, i.e.

$$g_{\mu\nu} = \Omega^{-2}\tilde{g}_{\mu\nu} \quad \text{where } \tilde{g}_{\mu\nu} \text{ smooth at } \mathscr{I}^+!$$

Another drawing of the conformal diagram is:



The shaded region is the COSMOLOGICAL CHART, in which the line-element takes the FRW-form (mostly used in cosmology)

$$ds^2 = -d\tau^2 + e^{2H\tau}(dx_1^2 + dx_2^2 + dx_3^2) \ .$$

DeSitter space is similar to Minkowski space in that it is a space of maximal symmetry, and that it has a well-behaved causal structure, so that equations like Klein-Gordon

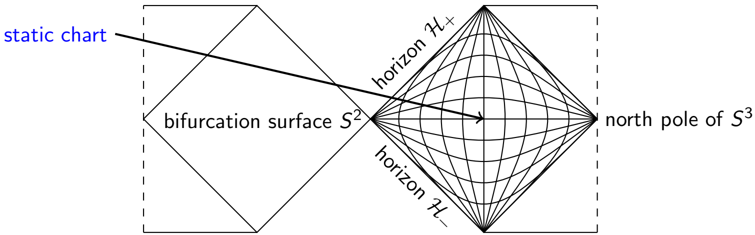
$$\left(\nabla^{\mu}\nabla_{\mu} - m^2\right)\phi = 0$$

have a well-posed initial value problem. But:

Unfamiliar features:

1. DeSitter has no analog of time-translation symmetry.
2. DeSitter fields like ϕ have no conserved Hamiltonian.

Except in static chart (shaded):



The free Klein-Gordon field is characterized by the OPE:

$$\phi(x)\phi(y) \sim \left\{ \frac{u}{\sigma + i0t} + \sum_n v_n \sigma^n \log(\sigma + i0t) \right\} \mathbf{1} + \phi^2(y) + \dots$$

where dots represent regular terms, and where u, v_n are the *Hadamard deWitt coefficients*. In deSitter, they can be given explicitly. The OPE implies the following:

Properties of correlators

1. Each $\langle \phi(X_1) \dots \phi(X_n) \rangle_\Psi$ satisfies KG-eqn.
2. $\langle \dots [\phi(X_1), \phi(X_2)] \dots \rangle_\Psi = i\Delta(X_1, X_2) \langle \dots \rangle_\Psi$.
3. The state is of "Hadamard form".

A state is "Gaussian" if it factorizes in the form

$$\langle \phi(X_1) \dots \phi(X_n) \rangle_\Psi = \sum_{\text{pairings}} \prod_{\text{pairs } ij} \langle \phi(X_i)\phi(X_j) \rangle_\Psi \ .$$

Existence of the OPE implies the following (essentially equivalent) properties:

- 2-pt function is analytic in $T^+ \times T^-$ where $T^+ = \{X \in dS^{\mathbb{C}} \mid \Im\, X \in \text{forward cone of } \mathbb{R}^5\}$
- Microlocal spectrum condition
- 2-pt function is "Hadamard".

These conditions still specify a **large class** of states. To obtain a more **specific** class of state one can impose:

Symmetry condition

E.g. 2-pt function should be $O(4,1)$-invariant.

The condition (+Gaussian) by itself specifies a **1-parameter family** of states [Allen 1986], the "alpha-vacua", when $m^2 > 0$. Together with the OPE, it specifies a **unique** state, called Bunch-Davies state.

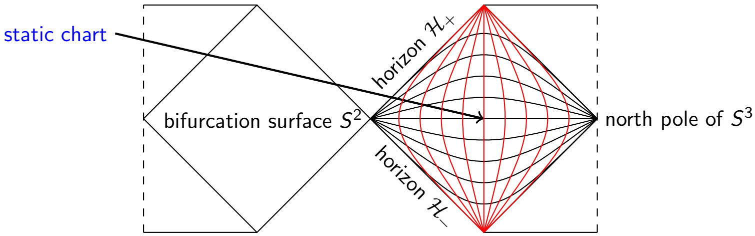
Bunch-Davies state

$$\langle \phi(X)\phi(Y)\rangle_{\mathrm{BD}} = \mathrm{cst.}\ {}_1F_2\left(-c, 3+c; 2; \frac{1+Z}{2}\right)$$

Here c is a dimensionless constant

$$c = -3/2 + (9/4 - m^2/H^2)^{1/2} < 0.$$

Although this state is called "vacuum", there is e.g. no useful notion of "1-particle" state built onto this vacuum. Actually, more like a thermal state in static chart!



Momentum space representation

The *positivity* of the BD-state follows from the "momentum space" representation [Bros, Epstein, Moschella, 1996-now]

$$\langle \phi(X)\phi(Y)\rangle_{\mathrm{BD}} \sim \int \frac{d^3\mathbf{k}}{\sqrt{\mathbf{k}^2+m^2}}(X\cdot\xi(\mathbf{k}))^{-c}(Y\cdot\xi(\mathbf{k}))^{3+c}$$

where $\xi(\mathbf{k})$ parametrizes a 3-dimensional subspace of $\mathbb{R}^5$ = analog of "mass-shell". Analogy in Minkowski space:

$$\langle \phi(x)\phi(y)\rangle_{\mathrm{M}} \sim \int \frac{d^3\mathbf{k}}{\sqrt{\mathbf{k}^2+m^2}}\; e^{ik(x-y)}\;.$$

However, this analogy **does not help in defining particles in a meaningful way** in dS-spacetime, in a way they "desintegrate" over distances larger than H^{-1}! BD-state is actually "thermal".

Interacting fields in deSitter

Fortunately, we **do not** want to calculate particle scattering amplitudes etc., but instead the n-pt. correlation functions $\langle\phi(x_1)\dots\phi(x_n)\rangle_\Psi$ in a theory with potential $V \sim m^2\phi^2 + \lambda\phi^4$! $\Rightarrow$ perturbative approach in λ.

Questions:

1. What states do we want to consider?
2. How to deal with the UV-divergences in a satisfactory way?
3. How to deal with (potential) IR-divergences?

Answers:

1. Probably any state would do, because we are interested in "late time" behavior of correlators, e.g. $\Psi =$ Bunch-Davies.
2. [S.H.-Wald 2001-2008, Brunetti et al. 2000, 1998, Radzikowski 1998]
3. Analytic continuation (for $m^2 > 0$) $dS_4 \to S^4$ and back.

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Thus, we are naturally led to consider the following expressions for the n-point fct's

$$\begin{aligned}&\langle \phi(X_1)\dots\phi(X_n)\rangle_{\mathrm{BD},\lambda}\\ = \;& \underset{X_i \to dS_4}{\text{anal. cont.}} \sum_V \frac{\lambda^V}{V!} \left\langle \prod_i \phi(X_i) \left(\int_{S^4} d\mu(Y)\, \phi^4(Y) \right)^V \right\rangle_{\mathrm{BD}}\end{aligned}$$

This is a perturbation expansion in powers λ^V. It is

1. IR-convergent, because all integrals are over compact S^4.
2. Analytic continuation should be possible. Resulting n-pt function $O(4,1)$-inv.
3. $d\mu(Y)$-integrations are UV-divergent $\rightarrow$ renormalization.

Applying the renormalization prescription of [S.H., Wald 2001, ..., Keller-Fredenhagen 2010]), and using a Mellin-Barnes representation one can show that the final result can be summarized in the parametric integral: [Hollands 2010]

$$\langle \prod \phi(X_i) \rangle_{\rm BD} = \underset{X_i \to dS_D}{\text{anal. cont.}} \sum_{\text{graphs } G} I_G(\{Z_{ij}\})$$

where

$$I_G(\{Z_{ij}\}) = \text{cst.} \left(\prod_l \int_{-i\infty}^{i\infty} \frac{dz_l}{\sin^2(\pi z_l)} \right) \int_\Delta [d\vec{\alpha}] \, \frac{\mathcal{F}^{-\epsilon}}{\mathcal{U}^{(D+1)/2-\epsilon}}$$
$$\times \prod_l \frac{\Gamma((D-1)/2 + i\rho + z_l)\Gamma((D-1)/2 - i\rho + z_l)}{\Gamma(1+z_l)\Gamma(D/2+z_l)}$$
$$\times \prod_{ij} \frac{\sin(\pi z_{ij})\Gamma(1+z_{ij})}{\pi \; \Gamma(z_i + (D+1)/2 + \epsilon)} \, \alpha_{ij}^{-1-z_{ij}} \prod_i \Re(\alpha_{i*})^{(D-1)/2+z_i+\epsilon} \; .$$

This formula is of the same general nature as integrals over "Feynman parameters" α_{ij} for ordinary loop integrals in Minkowski space, but somewhat more complicated. The integrals can be converted into sums using standard "tricks" involving the residue theorem. The final result is a function of the invariants Z_{ij}, i.e. basically the geodesic distances. One needs to perform finite renormalizations to get analytic dependence on m, H.

Explanation of formula:

- $[d\vec{\alpha}]$ measure on unit simplex.
- G Feynman graph w/ V interaction vertices
- $\mathcal{U}, \mathcal{F}$-homogeneous **graph polynomials** = determinants
- Close relation to "periods" in algebraic geometry
- I lines of graph G
- $c = -(D-1)/2 + i\rho$
- z_{ij} sum over z_l from lines connecting i, j, etc.

Quantum-no-hair/IR stability

The classical cosmic-no-hair-conjecture can be stated as saying that, away from black holes, a generic solution to e.g. Einstein-KG-Λ-theory behaves as

$$h_{\mu\nu} \sim O(\mathrm{e}^{2Ht}) , \quad \phi \sim O(\mathrm{e}^{-\alpha t}) \quad \text{as } t \to \infty,$$

where $h_{\mu\nu}$ is the deviation from exact deSitter, and some $\alpha > 0$. Thus, this is essentially the statement of exponential decay of perturbations. At the quantum level, it is natural expect that

Cosmic-no-hair/IR-stability

For a generic quantum state Ψ, we have

$$\langle \phi \rangle_\Psi \sim O(\mathrm{e}^{-\alpha t})$$

This is a version of cosmic-no-hair where quantum gravity effects are neglected.

1. By *generic state* one could mean e.g. states of the form (in Hilbert-space representation)

$$|\Psi\rangle = \sum \left(\prod_i \int f_i(X_i)\phi(X_i) \right) |\mathrm{BD}\rangle \ ,$$

where f_i is non-zero only inside a compact set. Such states are dense by Reeh-Schlieder thm. [Strohmayer, Verch, Wollenberg, 2000].

2. With such states, the cosmic-no-hair property is essentially equivalent to the decay of correlators in the BD-state,

$$\langle \phi(X_1) \ldots \phi(X_n) \rangle^c_{\mathrm{BD}} \sim O(Z_{ij}^{-\alpha}),$$

if pair of points is separated widely in time-like directions ($Z_{ij} \sim \mathrm{e}^{Ht_{ij}}$).

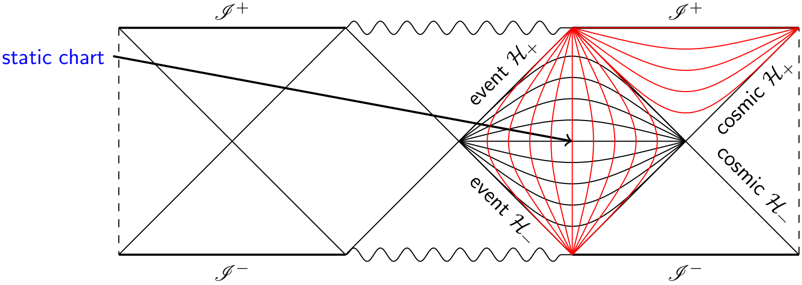
⇒ Generalized no-hair property for higher *n*-pt functions: Correlators of generic states exponentially approach BD at late times.

The exponential decay

$$\langle \phi(X_1)\dots\phi(X_n)\rangle^c_{\mathrm{BD}} \sim O(Z_{ij}^{-\alpha}),$$

has very recently been proven to all orders in perturbation theory in $\lambda\phi^4$-theory (with $m^2 > 0$) independently by

1. Marolf & Morris, using an iterative argument, and by
2. Myself, using the parametric representation and a famous "forest formula" of Brydges-Kennedy.
3. The generalization of this result to (a) $m^2 = 0$ or (b) Schwarzschild deSitter would be important!



Conclusions

In this talk I have:

1. Outlined what are typical questions in QFT on curved spacetime, and what are not sensible questions.
2. Outlined general framework and theorems in curved spacetime (PCT)
3. Told you a parametric formula for Feynman diagrams in deSitter spacetime
4. Told you that massive scalar fields on deSitter are IR-stable/satisfy a quantum version of the cosmic-no-hair theorem. see however [Polyakov 2009]

Further contributors to these topics include

1. General theorems: Brunetti, Fewster, Frohlich & Birke, Fredenhagen, Haag, Kay, Radzikowski, Verch, Wald,...
2. deSitter: Anderson, Balasubramanian, Allen, Einhorn, Friedrich, Higuchi, Jacobson, Jaekel & Gerard, Larsen, Minic, Mottola, Moretti, Dappiaggi & Pinamonti, Randall, Strominger, Spradlin, Tsamis, Verlinde, Volovic, Woodard,...

Conclusions

In this talk I have not covered some other interesting topics such as:

1. Quantum inequalities, quantum interest etc. Fewster, Ford, Roman, ...
2. Entanglement entropy Takayanagy, ...
3. Renormalization of non-abelian gauge theories S.H.
4. Renormalization via flow equations Kopper, Muller
5. Holographic renormalization
6. Casimir effect, Hawking effect, Unruh effect