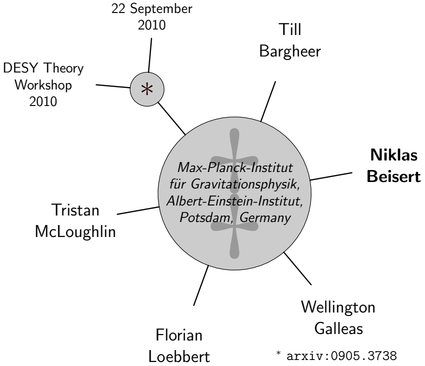
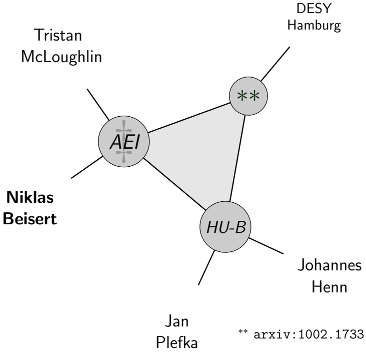
Integrability for Scattering Amplitudes in Planar $\mathcal{N} = 4$ super Yang-Mills*,**



* arxiv:0905.3738



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Remarkable $\mathcal{N} = 4$ Super Yang–Mills

$\mathrm{U}(N_\mathrm{c})$ gauge field $\mathcal{A}_\mu$, 4 fermions Ψ^a_α, 6 scalars Φ_m

$$S_{\mathcal{N}=4} \sim \frac{N_\mathrm{c}}{\lambda} \int \frac{d^4x}{4\pi^2} \,\mathrm{Tr}\Big(\tfrac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \tfrac{1}{2}(\mathcal{D}_\mu\Phi_m)^2 - \tfrac{1}{4}[\Phi_m,\Phi_n]^2 + \ldots\Big).$$

- Unique action, three unrenormalised couplings λ, N_c, θ_top.
- Exact superconformal symmetry $\mathfrak{psu}(2,2|4)$.
- And some mysterious features: AdS/CFT, integrability, dualities, . . .

Magic in the Planar Limit:

- Integrability in the planar limit: $\mathfrak{psu}(2,2|4)$ Yangian.
- Planar anomalous dimensions (presumably/largely) solved.
- Simplifications for planar scattering, dual conformal symmetry.
- Novel scattering tools: Twistors, CSW/BCF, Graßmannian, TBA, ...

I. Overview:

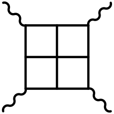
Gluon Amplitudes, AdS/CFT and Integrability

Planar Scattering Amplitudes

Intriguing result in $\mathcal{N}=4$ SYM in the planar limit $N_\text{c}\to\infty$:

Four-gluon scattering amplitude obeys BDS relation [Anastasiou, Bern, Dixon, Kosower][Bern, Dixon, Smirnov]

$$A(p,\lambda)\simeq A^{(0)}(p)\exp\Big(2D_\text{cusp}(\lambda)M^{(1)}(p)+F(p,\lambda)\Big)\,.$$



Only required data: • tree level, • one loop, • cusp dimension.

- No finite remainder function $F(p,\lambda)=0$.

Scattering amplitudes constructible by unitarity and suitable ansatz.
Verified BDS relation at $\mathcal{O}(\lambda^4)$ with [Bern, Dixon, Smirnov][Bern, Czakon, Dixon, Kosower, Smirnov]

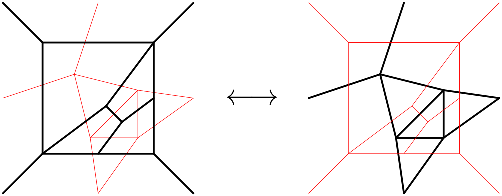
$$D_\text{cusp}(\lambda)=\frac{1}{2}\,\frac{\lambda}{\pi^2}-\frac{1}{96}\,\frac{\lambda^2}{\pi^2}+\frac{11}{23040}\,\frac{\lambda^3}{\pi^2}-\Big(\frac{73}{2580480}+\frac{\zeta(3)^2}{1024\pi^6}\Big)\,\frac{\lambda^4}{\pi^2}\pm\ldots\,.$$

Non-trivial finite remainder for $n\geq 6$. [Alday, Maldacena][Drummond, Henn, Korchemsky, Sokatchev][Bartels, Lipatov, Sabio Vera][Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich]

Simplicity and Dual Conformal Symmetry

Planar amplitudes and integrals simpler than expected:

- No bubbles, no triangles, typically scalar boxes (also non-planar).
- Similarity of momentum and position space propagators in $D = 4$.



- Dual amplitudes and integrals are conformal. [Drummond Korchemsky Sokatchev]...
- Dual superconformal symmetry of superspace amplitude. [Drummond, Henn Korchemsky Sokatchev]

Strings:

- Self-duality of superstrings requires also fermionic T-duality. [Berkovits Maldacena]
- Dual superconformal symmetry $\widehat{=}$ symmetry of T-dual model.
- Dual superconformal symmetry allows $F(p, \lambda)$ only for $n \geq 6$ legs.

Sketch of Scattering Amplitudes

Structure of S-matrix elements (in regularised theory):

$$\mathcal{A} \simeq \sum \text{Colour} \times \text{Polarisation} \times \text{Scalar Loop Integrals}.$$

Scalar factor contains IR divergences (massless particles)

$$\text{Scalar Factor} \simeq \exp(\text{IR Singularity} + \text{Finite Remainder}).$$

Same cusp dimension $D_{\text{cusp}}(\lambda)$ in IR singularity & integrable system:

- How to apply integrability to planar scattering amplitudes?
- Can one also compute remainder function $F(p, \lambda)$?
- Relation between (dual) superconformal symmetry and integrability?

Modern Efficient Tools related to integrability:

- Polarisation: Graßmannian. [Arkani-Hamed, Cachazo, Cheung, Kaplan] [Bullimore, Mason, Skinner] [Kaplan 0912.0957] [Arkani-Hamed, Bourjaily, Cachazo, Trnka]
- Scalar Factor: TBA. [Alday, Maldacena] [Alday, Gaiotto, Maldacena] [Alday, Maldacena, Sever, Vieira]

Outline

Symmetries of Scattering Amplitudes (S-matrix):

- Understand symmetries of S-matrix.
- Apply symmetries to (fully?) constrain S-matrix.

How to treat (super)conformal symmetry of the S-matrix in $\mathcal{N} = 4$ SYM?

Concretely

- Structure of symmetries: superconformal and Yangian
- Free symmetries
- Symmetries at tree level
- Symmetries at one loop
- ...

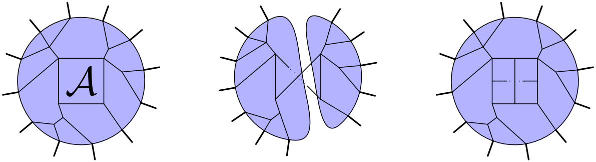
Ultimately

- Perturbative symmetries increasingly messy. Not so useful.
- Symmetries are non-perturbative!(?) Exploit them there!

II. Free Symmetries

Scattering Amplitudes

Colour-ordered scattering amplitudes (1-trace, 2-trace, genus-1):



Legs: Field Ω combines on-shell gluons Γ, fermions Ψ & scalars Φ:

$$\Omega(\lambda,\tilde{\lambda},\bar{\eta}) = \Gamma(\lambda,\tilde{\lambda}) + \bar{\eta}^a \Psi_a(\lambda,\tilde{\lambda}) + \tfrac{1}{2}\bar{\eta}^a\bar{\eta}^b \Phi_{ab}(\lambda,\tilde{\lambda}) + \ldots .$$

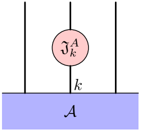
Amplitude $\mathcal{A}(\Lambda_1,\ldots,\Lambda_n)$ on spinor helicity superspace $\Lambda = (\lambda,\tilde{\lambda},\bar{\eta})$

$$p^{\beta\dot{\alpha}} = \lambda^\beta \tilde{\lambda}^{\dot{\alpha}}, \qquad q^{\alpha b} = \lambda^\beta \bar{\eta}^a.$$

Free Superconformal Symmetry

Free representation $\mathfrak{J}$ of superconformal algebra $\mathfrak{psu}(2,2|4)$:

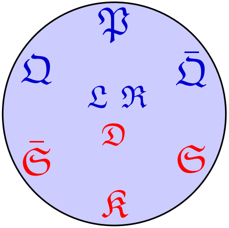
$$\mathfrak{J} = \sum_{k=1}^{n}$$



- Single-leg representation $\mathfrak{J}_k$: multiplet of free on-shell fields Ω.
- Symmetry: Amplitudes are invariant under $\mathfrak{J}$.
- Representation independent of colour structure.

Algebra Generators:

- super-Poincaré: $\mathfrak{P}$, $\mathfrak{L}$, $\mathfrak{Q}$, $\bar{\mathfrak{Q}}$, $\mathfrak{R}$
- superconformal: $\mathfrak{K}$, $\mathfrak{D}$, $\mathfrak{S}$, $\bar{\mathfrak{S}}$

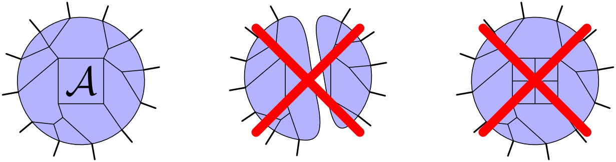


Invariance of tree (N*MHV) amplitudes:

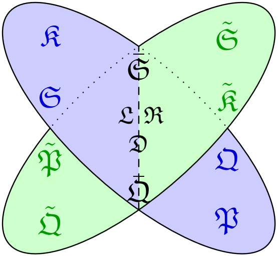
- $\mathfrak{P}$, $\mathfrak{Q}$: manifest through delta functions,
- $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{D}$: weight and index contractions,
- $\bar{\mathfrak{Q}}$, $\bar{\mathfrak{S}}$: medium difficulty,
- $\mathfrak{K}$, $\mathfrak{S}$: hardest.

Dual Superconformal Symmetry

Remarkable features of disk amplitudes (single-trace, large-N_c):



- Simplifications: BDS formula.
- Only particular integrals appear. [Drummond Korchemsky Sokatchev]
- Dual superconformal symmetry! [Drummond, Henn Korchemsky Sokatchev]
- Another $\mathfrak{psu}(2,2|4)$:
  $\mathfrak{L}$, $\mathfrak{R}$, $\mathfrak{D}$, $\bar{\mathfrak{Q}}$, $\bar{\mathfrak{S}}$: shared with conventional,
  $\tilde{\mathfrak{P}}$, $\tilde{\mathfrak{Q}}$: trivial new generators,
  $\tilde{\mathfrak{K}}$, $\tilde{\mathfrak{S}}$: non-trivial new generators.



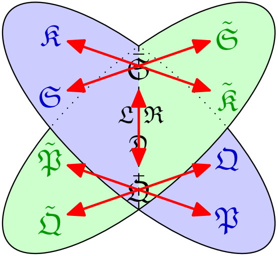
T-Self-Duality in String Theory

Detour: Consider string theory picture at strong coupling.

- Strings propagate on $AdS_5 \times S^5$ superspace.
- Background is coset space $\mathrm{PSU}(2,2|4)/\mathrm{Sp}(2) \times \mathrm{Sp}(1,1)$.
- Isometries of background are Noether symmetries: $\mathfrak{psu}(2,2|4)$.

T-duality transformation: [Alday Maldacena][Berkovits Maldacena]

- 4 bosonic + 8 fermionic T-dualities.
- terms at worldsheet boundaries: planar!
- maps $AdS_5 \times S^5$ string model to itself: self-duality!
- maps isometries to dual isometries: dual superconformal symmetry



String Theory Integrability

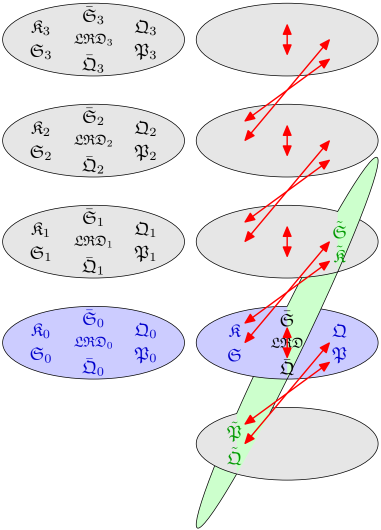
Integrability enhances conserved charges:

$$Q(z) = z\mathcal{Q}_0 + z^2 Q_1 + z^3 Q_2 + \ldots$$

- Q_0 are Noether charges.
- Q_k form (half) loop algebra.
- ∞-dimensional hidden symmetries.

T-self-duality [NB, Ricci Tseytlin, Wolf][Berkovits Maldacena][NB 0903.0609]

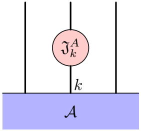
- maps loop algebra to itself.
- It shows that conventional & dual superconformal symmetry close into loop algebra.
- Quantum algebra called Yangian.

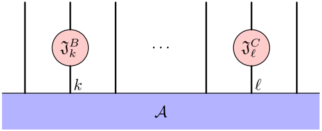


Yangian Symmetry

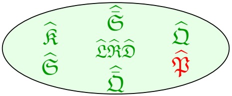
Back to scattering amplitudes in $\mathcal{N}=4$ SYM:

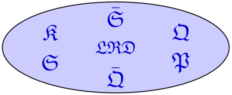
Free representation $\mathfrak{J}=\mathfrak{J}_0$, $\widehat{\mathfrak{J}}=\mathfrak{J}_1$ of $\mathfrak{psu}(2,2|4)$ Yangian: [Drummond Henn Plefka]

$\mathfrak{J}^A = \sum_{k=1}^{n}$ 

$\widehat{\mathfrak{J}}^A = F^A_{BC} \sum_{k<\ell=1}^{n}$ 

- Yangian Symmetry:
  Amplitudes are invariant under $\mathfrak{J}$, $\widehat{\mathfrak{J}}$. [Drummond, Henn Korchemsky Sokatchev]
- compatible with cyclic structure (exceptional)!
- $\mathfrak{J}$ & $\widehat{\mathfrak{J}}$ all one needs to know.
  In fact, only $\mathfrak{J}$ and $\widehat{\mathfrak{P}}=\tilde{\mathfrak{K}}$ sufficient.
- Representation requires planar limit & ordering; depends on colour structure.
- Planar amplitudes are integrable!





Meaning of Integrability

Wait a minute! Integrability makes scattering in $D > 2$ trivial!?!

- Only if integrability refers to local conserved charges.
- Here conserved charges are local in (planar) colour space: String!

Integrability is symmetry enhancement. Properties of S-matrix:

- unitarity
- (super) Poincare invariance
- (super) conformal invariance
- Yangian: Infinite-dimensional algebra (one additional constraint).

Integrability means (pragmatic definition):

- hidden symmetry constrains S-matrix uniquely.
- can calculate S-matrix efficiently: Graßmannian, TBA.

III. Superconformal Anomaly at Tree Level

Nitpicking

Graßmannian generates free Yangian invariants. [Drummond Ferro] [Korchemsky Sokatchev]

Tree level S-matrix is suitable linear combination. Which? [Drummond Henn]

Invariants?

- Individual invariants have spurious singularities. [Hodges 0905.1473]
- Individual invariants have wrong collinear behaviour. [Korchemsky Sokatchev]
- "Invariants" actually not exactly invariant. [Cachazo Svrcek Witten]
  Free symmetries have distributional anomalies!
- Ignore at tree level $\Rightarrow$ hits you hard at loops.
  Anomaly smeared by loop integration.
- Repair anomaly by deformed representation. [Bargheer, NB, Galleas Loebbert, McLoughlin] [NB, Henn McLoughlin, Plefka]

Invariant!

- There can be only one invariant: the S-matrix.
- S-matrix assembled from almost-invariants.

Collinear Anomaly

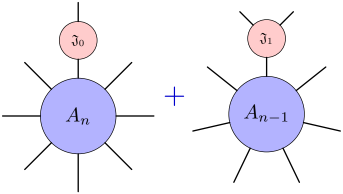
Tree amplitude has poles when particle momenta become collinear

$$\mathcal{A} \sim \langle k, k+1 \rangle^{-1}.$$

Conformal symmetry sensitive to poles. Distributional anomaly

$$\bar{\mathfrak{J}}\mathcal{A} \sim \delta^2(\langle k, k+1 \rangle).$$

Compensate by deformation of conformal representation [Bargheer, NB, Galleas Loebbert, McLoughlin]

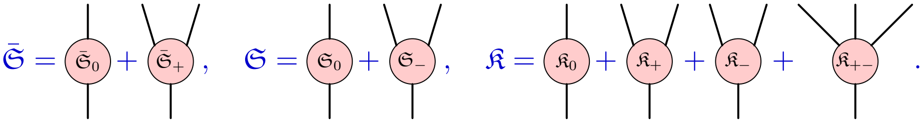


$$+ \quad = 0.$$

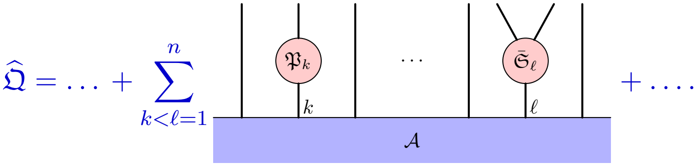
Only complete S-matrix (not individual amplitudes) invariant!

Classical Representation

We find the following corrections for representation of $\mathfrak{S}$, $\bar{\mathfrak{S}}$ and $\mathfrak{K}$



Similar deformations expected for classical Yangian representation, e.g.

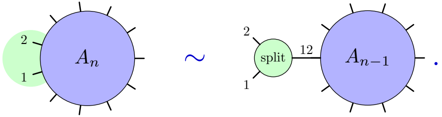


To be done:

- Does the deformation annihilate all tree amplitudes? Yes.
- Is it a consistent representation of superconformal symmetry? Yes.
- What does it mean? Conformal symmetry & physical asymptotic states.

Invariance of Tree Amplitudes

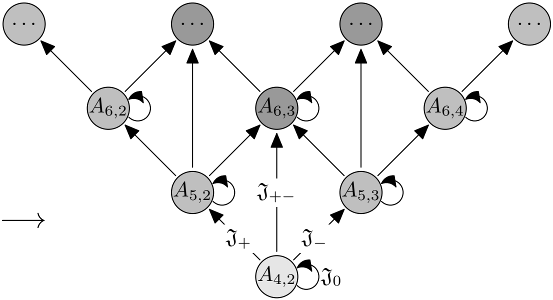
Collinear limit is universal for all (tree) amplitudes: splitting function



Follows e.g. from recursion relation by inheritance.

Exact invariance of all tree amplitudes: [Bargheer, NB, Galleas Loebbert, McLoughlin]

- Collinear singularities universal, same as for MHV.
- No anomalies from multi-particle singularities.
- Structure of cancellations $\longrightarrow$ $A_{n,k}$: n-leg N^{k-2}MHV.



Uniqueness

All tree amplitudes have been constructed by recursion relations [Drummond Henn]

$$A_{n,k}(p) = A_n^{\mathrm{MHV}}(p) \sum_\alpha c_\alpha R_\alpha(p).$$

Each R is (almost) invariant under the free Yangian representation.
How to obtain the correct physical linear combination $c_\alpha = 1$?

- Demand absense of spurious singularities or equivalently [Hodges 0905.1473]
- demand correct collinear behaviour. [Korchemsky Sokatchev]

Deformed representation ensures correct collinear behaviour. [Bargheer, NB, Galleas Loebbert, McLoughlin]
Therefore symmetry alone fixes correct linear combination $c_\alpha = 1$!

Very important for construction by symmetry at higher loops:

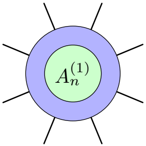
- Adding any invariant respects symmetry: ambiguity!
- Adding tree level amounts to an overall factor; okay.

Can symmetry fix planar amplitude completely (non)perturbatively?

IV. Superconformal Symmetry of Loop Amplitudes

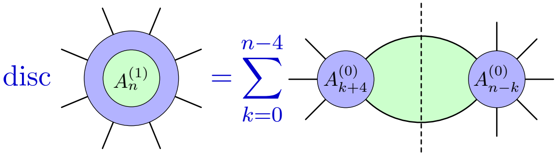
Symmetry of Unitarity Cuts

What about conformal symmetry for loop amplitudes?



- Particles in loop are off-shell.
- Conformal symmetry applies to on-shell particles only.
- How to determine loop anomaly?

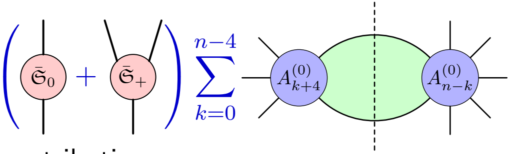
Idea: Consider unitarity cut of loop amplitude [Korchemsky Sokatchev] [Brandhuber Heslop Travaglini] [Sever Vieira]



- External and loop particles on shell!
- Only tree amplitudes used.
- Can act with symmetry to determine anomaly of cut.
- Use dispersion integral to reconstruct anomaly of loop later.

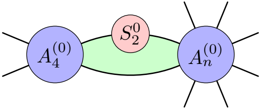
Anomaly of Unitarity Cuts

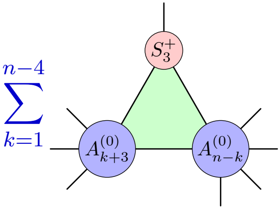
Act with superconformal boost on unitarity cut using invariance of trees

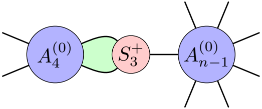


Three remaining contributions:

[NB, Henn McLoughlin, Plefka]







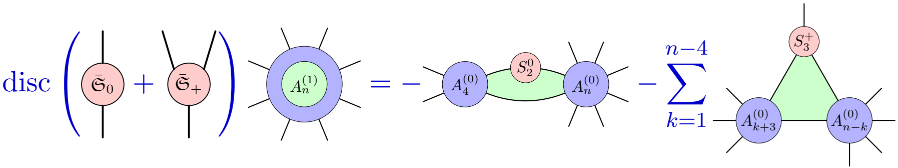
- from divergent measure
- no momentum transfer

- integral localised
- finite, rational

- one-loop splitting
- ill-defined→zero

One-Loop Anomaly

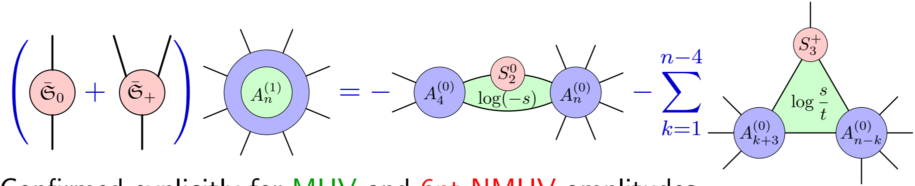
The anomaly of unitarity cuts reads



How to promote it to the loop anomaly?

[NB, Henn McLoughlin, Plefka]

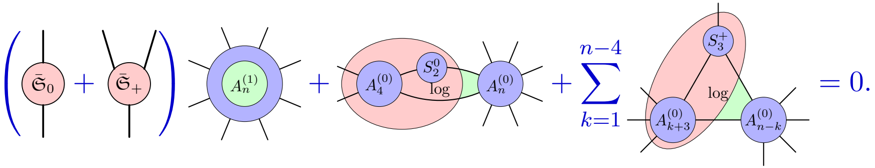
- All integrals are fully localised, cut anomaly is rational.
- Multiply by logs of cut invariants to reproduce cut discontinuity.



Confirmed explicitly for MHV and 6pt NMHV amplitudes.

One-Loop Invariance

Can rewrite loop anomaly as invariance statement: [NB, Henn McLoughlin, Plefka]



One-loop superconformal invariance of scattering amplitudes

$$\bar{\mathfrak{S}}^{(0)}\mathcal{A}^{(1)} + \bar{\mathfrak{S}}^{(1)}\mathcal{A}^{(0)} = 0.$$

- Different from earlier proposal: [Sever Vieira]
  Uses only on-shell amplitudes, compatible with regulator.
- Tree anomalies: collinearities $(1 \to 2)$.
- Loop anomalies: measure $(2 \to 2)$, loop collinearities $(2 \to k)$.

Superconformal Algebra

Do the loop corrections of the algebra close?

- Measure contributions take a simple universal form $\Rightarrow$ closure

$$\mathfrak{J}^{(1)}_{2\to 2} \sim [X_{2\to 2}, \mathfrak{J}^{(0)}], \qquad X_{2\to 2} = \sum_{k=1}^{n} \frac{1}{\epsilon^2} \left(\frac{s_{k,k+1}}{-\mu^2} \right)^{-\epsilon}.$$

- Corrections $\bar{\mathfrak{S}}^{(1)}_{2\to k}$ are invariant under super-Poincaré.

Rest of superconformal algebra is not yet clear:

- Some indications that algebra may not close properly.
- Deformations for $\bar{\mathfrak{S}}^{(1)}_{2\to k}$ not uniquely determined by amplitudes.
- Closure on which class of functions?
  Only amplitude itself: only singlet representation consistent?
- Can superconformal algebra be defined at loop level?

In any case, we understand the one-loop superconformal anomaly.

V. Conclusions

Conclusions

⋆ **Superconformal Symmetry at Tree Level**

- Tree amplitudes almost invariant under free superconformal symmetry.
- Invariance violated for singular configurations: Collinear momenta.
- Transformations can be corrected to make trees fully invariant.
- Dynamic corrections requires, changes number of legs.
- Superconformal algebra closes onto gauge transformations.
- Yangian appears to lead to a unique invariant $\Rightarrow$ the S-matrix.

⋆ **Superconformal Symmetry at One Loop**

- Transformations can be corrected to make loops fully invariant.

⋆ **Open Problems**

- How does the algebra at loop level close?
- What's new at two loops?
- What about conformal inversions?