New differential equations for on-shell loop integrals

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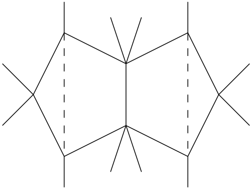
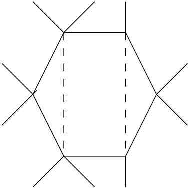
based on

F. Alday, J. H., J. Plefka and T. Schuster, arXiv:0908.0684
J. H., S. Naculich, H. Schnitzer and M. Spradlin, arXiv:1001.1358, arXiv:1004.5381
J. Drummond, J. H., arXiv:1008.2965
J. Drummond, J. H., J. Trnka, to appear

DESY 2010, Hamburg, September 22

Differential equations for on-shell loop integrals

- established method [Kotikov 1991; Gehrmann, Remiddi, 1999; ...]
  - differentiate w.r.t. masses or momenta (i.e. Mandelstam invariants)
  - first-order differential equations
  - apply to a set of master integrals and family of reduced integrals (reduction identities have to be known)
- this talk: new method [Drummond, Henn, Trnka, to appear]
  - applies to certain infinite classes of loop integrals (motivated by $\mathcal{N} = 4$ SYM)
  - apply to one integral at a time (no integral reduction needed)
  - second-order operators reduce the loop order by one $\rightarrow$ iterative structure

$\mathcal{D}^{(2)}$  $=$ 

New differential equations for on-shell loop integrals

- Part 1: planar scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills
  - motivation
  - masses to regulate IR divergences
  - (extended) dual conformal symmetry
  - implications for integral basis at higher loops
- Part 2: New differential equations for loop integrals
  - Differential equations for off-shell integrals
  - Differential equations for on-shell integrals

Part 1: planar scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills

Scattering amplitudes in $\mathcal{N} = 4$ SYM - motivation

supersymmetric YM as a tool for QCD

1. perturbatively, the theories are very similar
   $\Rightarrow$ certain tree-level amplitudes identical in both theories
   $\Rightarrow$ at one loop, susy decomposition:
   $$A_g = \underbrace{(A_g + 4A_f + 3A_s)}_{\mathcal{N}=4} - 4\underbrace{(A_f + A_s)}_{\mathcal{N}=1} + A_s$$
2. develop and test new methods in $\mathcal{N} = 4$ SYM
   $\Rightarrow$ e.g. recursion relations for tree amplitudes, (generalized) unitarity
   $\Rightarrow$ application to QCD

Blackhat uses analytic tree-level amplitudes obtained from $\mathcal{N} = 4$ SYM!

[Blackhat collaboration: Berger et al] [Drummond, Henn, Korchemsky, Sokatchev 2008; Drummond, Henn 2008]

Ambitious goals and prospects in $\mathcal{N} = 4$ SYM

1. Discover and understand new hidden symmetries (e.g. dual conformal symmetry)
2. Compute amplitudes for arbitrary number of legs and/or loops?!
3. Test AdS/CFT

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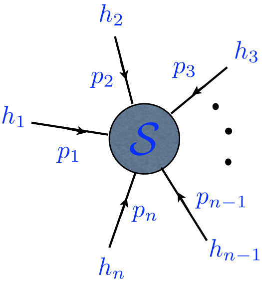
Scattering amplitudes in $\mathcal{N} = 4$ SYM

- n-particle scattering amplitude



  helicity: $h_i = 0$ scalar, $h_i = \pm 1$ gluon, $h_i = \pm\frac{1}{2}$ gluino

- color structure

$$A_n(\{p_i, h_i, a_i\}) = \sum_{\sigma \in S_n/Z_n} \mathrm{tr}[t^{a_1} \ldots t^{a_n}] \times \mathcal{A}_n(\{p_{\sigma_1}, h_{\sigma_1}\}, \ldots, \{p_{\sigma_1}, h_{\sigma_1}\})$$

- planar limit $\lambda = g^2 N$ fixed, $N \to \infty$
- IR divergences (well-understood) due to massless particles
  use e.g. dimensional regularization;
  this talk: use Higgs masses as a regulator

Reminder: Dual conformal symmetry

- dual coordinates x_i^μ: [Broadhurst 1991; Lipatov 1993]

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu$$

- Planar MHV amplitudes have $SO(4,2)$ symmetry in the dual x_i space

  [Drummond, J.H., Smirnov, Sokatchev, 2006; Drummond, J.H., Korchemsky, Sokatchev, 2007]

- Can be extended to dual superconformal symmetry applicable to MHV and non-MHV amplitudes [Drummond, J.H., Korchemsky, Sokatchev, 2008]
- Conventional + dual superconformal → Yangian symmetry [Drummond, J.H., Plefka, 2009]

$$J^{(0)A}{}_B = \sum_i Z_i^A \frac{\partial}{\partial Z_i^B}\,, \qquad J^{(1)A}{}_B = -\sum_{i<j} Z_i^A Z_j^C \frac{\partial}{\partial Z_i^C} \frac{\partial}{\partial Z_j^B} - (i \leftrightarrow j)$$

  related references: [Beisert et al; Korchemsky, Sokatchev; Arkani-Hamed et al; Mason, Skinner; Drummond, Ferro]

  → talk by N. Beisert at this conference

- (unregulated) all-loop planar *integrand* has Yangian symmetry [Arkani-Hamed et al, 2010]

Planar amplitudes on the Coulomb branch of $\mathcal{N} = 4$ Super Yang-Mills

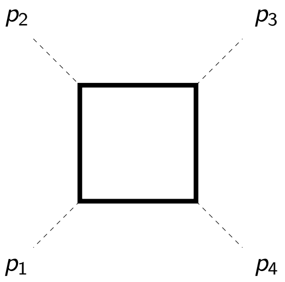
- $U(N+M) \to U(N) \times U(M)$

[Alday, Maldacena, 2007; Kawai, Suyama, 2007; Schabinger, 2008; Sever, McGreevy, 2008]

[Alday, J.H., Plefka, Schuster, 2009]

$\rightarrow$ leads to massive particles

- scatter massless $U(M)$ particles
- $N \gg M$: only allow loops in N-part of $U(N+M)$
  $\rightarrow$ renders amplitudes IR finite
- e.g. colour-ordered four-point one-loop amplitude

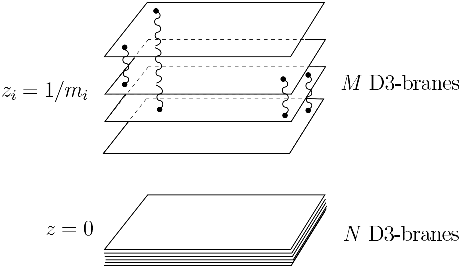


$$M^{(1)}(m^2/s, m^2/t)\,, \qquad s = (p_1+p_2)^2\,, \quad t = (p_2+p_3)^2$$

String theory motivation

- string picture:

  Alday, Maldacena



(a)

- bosonic + fermionic T-duality is relevant [Alday, Maldacena, 2007; Berkovits, Maldacena, 2008]
- isometries of AdS_5 in T-dual theory

$$J_{-1,4} = m\partial_m + x^\mu \partial_\mu = \hat{D}$$

$$J_{4,\mu} - J_{-1,\mu} = \partial_\mu = \hat{P}_\mu$$

$$J_{4,\mu} + J_{-1,\mu} = 2y_\mu(x_\nu \partial^\nu + m\partial_m) - (x^2 + m^2)\partial_\mu = \hat{K}_\mu$$

- Expectation: Amplitudes regulated by Higgs masses should be invariant exactly under extended dual conformal symmetry $\hat{K}_\mu$ and $\hat{D}$! [Alday, J.H., Plefka, Schuster, 2009]

[similar ideas used in Jevicki, Kazama, Yoneda, 1998]

Implications for higher loop integral basis

- basis of loop integrals in $\mathcal{N}=4$ SYM constrained by dual conformal symmetry?

  [Drummond, J.H., Smirnov, Sokatchev, 2006; Bern, Czakon, Dixon, Kosower, Smirnov, 2006; Bern, Carrasco, Johansson, Kosower, 2007]

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  [Spradlin, Volovich, Wen, 2008]

- it seems reasonable to speculate that [J.H., Naculich, Schnitzer, Spradlin, 2010]

$$M_n = 1 + \sum_{\mathcal{I}} a^{L(\mathcal{I})}\, c(\mathcal{I})\, \mathcal{I}\,,$$

  where: coupling a, loop order $L(\mathcal{I})$

  coefficients $c(\mathcal{I}) \Rightarrow$ compute by (generalized) unitarity

  integrals $\mathcal{I} \Rightarrow$ restricted set of extended dual conformal integrals

- absence of triangles at one loop [Boels, 2010; also: Schabinger, 2008]
- additional constraints from expected IR structure [Korchemsky, Sterman,...]

$$M_n = \exp\left[-\frac{1}{8}\Gamma_{\text{cusp}}(a)\sum_i \log^2\frac{s_i}{m^2} - \frac{1}{2}\tilde{G}_0(a)\sum_i \log\frac{s_i}{m^2} + \mathcal{O}(\log^0 m^2)\right]$$

- insights from analytic structure for generic m^2, and Regge limit(s)?
- further constraints from the (broken) conventional conformal symmetry?

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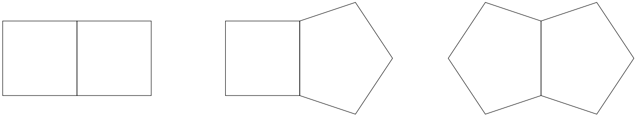
Application to two-loop integrals/amplitudes

- expected dual conformal integrals:



see e.g. six-point two-loop MHV case (in dimensional regularization)

[Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich, 2008]

and for higher-point amplitudes [Vergu, 2009]

- integrals can be evaluated straightforwardly (numerically) in mass regularization

[Henn, Naculich, Schnitzer, Spradlin 2010; Drummond, Henn 2010]

analytic result for limit of six-point remainder function: [Drummond, Henn 2010]

$$\lim_{u\to 0} \mathcal{R}_6^{(2)}(u,u,u) = \frac{\pi^2}{8}\log^2 u + \frac{17\pi^4}{1440} + \mathcal{O}(u)$$

$$\lim_{u\to \infty} \mathcal{R}_6^{(2)}(u,u,u) = -\frac{\pi^4}{144} + \mathcal{O}(1/u)$$

numerical result: [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich, 2008]

Summary of first part: properties of the Higgs regulator

Conceptual advantages

- natural from AdS/CFT viewpoint
- makes dual conformal symmetry exact
- restricts integral basis
- masses have physical interpretation

Practical advantages

- higher loop orders of amplitudes easy to compute
  e.g. no $\mathcal{O}(\epsilon) \times 1/\epsilon = \mathcal{O}(1)$ problems as in dimensional regularization
- Regge limit can be computed systematically
  e.g. LL and NLL computed to all orders [J.H., Naculich, Schnitzer, Spradlin, 2010]
- previously hard/impossible computations seem accessible
  e.g. two-loop amplitudes with $n \geq 5$ external particles
  [J.H., Naculich, Schnitzer, Spradlin, 2010; Drummond, J.H., 2010]

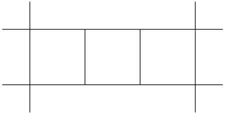
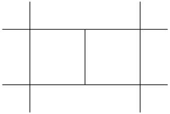
Part 2: differential equations

Off-shell ladder integrals

- Basic idea: use Laplace equation [Drummond, J.H., Smirnov, Sokatchev 2006]

$$\Box_1 \frac{1}{(x_1 - x_i)^2} \propto \delta^{(4)}(x_1^\mu - x_i^\mu)$$

- reduces loop order by one:

$\Box_1$  $\propto$ 

- iterative equations:

$$y\partial_y \, z\partial_z \, \Phi_L(y,z) = \Phi_{L-1}(y,z)$$

- infinite class of integrals known explicitly [Davydychev, Usyukina, 1993; Isaev 2003]

$$\Phi_L(y,z) = \sum_{r=0}^{L} \frac{(-1)^{r+L}(2L-r)!}{r!(L-r)!L!} \log^r(yz)\big(\mathrm{Li}_{2L-r}(y) - \mathrm{Li}_{2L-r}(z)\big)$$

Momentum twistor variables

- standard spinor helicity formalism: $p_i^{\alpha\dot{\alpha}} = \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}$
- dual coordinates: $p_i^{\mu} = x_i^{\mu} - x_{i+1}^{\mu}$
  relevant for dual conformal symmetry
- dual points $x_i^{\mu} \leftrightarrow Z_{i-1}^{[A}Z_i^{B]}$ lines in momentum twistors space [Hodges 2009]

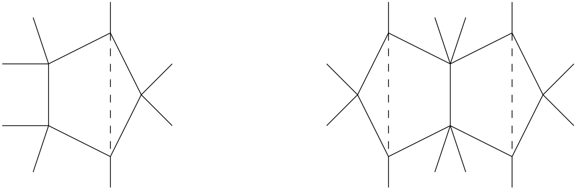
$$Z_i^A = (\lambda_i^{\alpha}, x_i^{\dot{\alpha}\beta}\lambda_{i\,\beta}), \qquad (x_i - x_j)^2 = \frac{(Z_{i-1}Z_iZ_{j-1}Z_j)}{\langle i-1\,i\rangle\langle j-1\,j\rangle}$$

where $(1234) = \epsilon_{ABCD}Z_1^AZ_2^BZ_3^CZ_4^D$ and $\langle 12\rangle = \lambda_1^{\alpha}\lambda_{2\,\alpha}$
important: solve on-shell and momentum conservation

- application to one-loop box integrals [Hodges 2010; Mason, Skinner 2010]
- integrals with special twistor numerators [Arkani-Hamed et al 2010; Drummond, J.H. 2010]

Example: two-loop MHV amplitudes

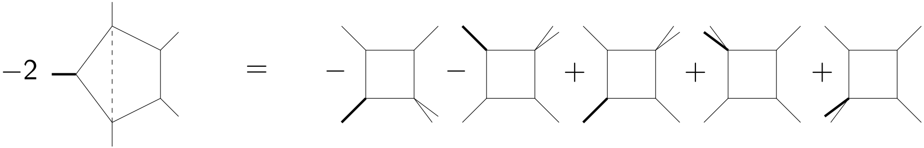
- representation of two-loop MHV amplitude of can be written in terms of one integral topology [Arkani-Hamed et al., 2010] [Drummond, Henn, Trnka, to appear]



numerator of pentagon $(Z_a Z_b Z_1 Z_3)$ where $Z_a Z_b \leftrightarrow$ integration point

- these integrals satisfy differential equations [Drummond, Henn, Trnka, to appear]
- relation to standard integrals: trivial integral reduction identities

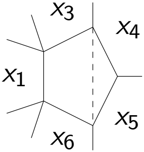
[Drummond, Henn, 2010; Arkani-Hamed et al, 2010]



Differential equations for on-shell integrals

- Example: one-loop pentagon with off-shell leg [Drummond, Henn, Trnka, to appear]

$$I^{\text{pentagon}} :=$$



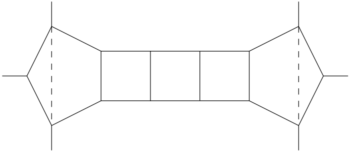
  - could use Laplace operator:

$$\Box_1 \, I^{\text{pentagon}} \propto 1$$

  - momentum twistor operator does the same job!

$$Z_3 \cdot \frac{\partial}{\partial Z_4} Z_2 \cdot \frac{\partial}{\partial Z_3} \, I^{\text{pentagon}} \propto 1$$

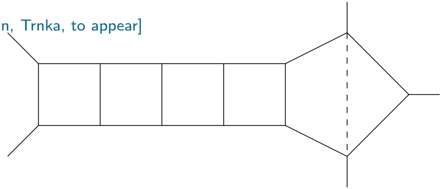
  - obtain iterative relations for any integral with this sub-topology, e.g.



Note: Laplace operator not applicable here!

Example

- five-point example: [Drummond, Henn, Trnka, to appear]
  integral regulated by m^2



- differential equations (variables $y_1 = x_{35}^2/x_{13}^2, y_2 = x_{25}^2/x_{24}^2, y_3 = x_{14}^2/m^2$)

$$y_1\partial_{y_1}\, y_2\partial_{y_2}\, \Psi^{(L)}(y_1, y_2, y_3) = \Psi^{(L-1)}(y_1, y_2, y_3) + \mathcal{O}(m^2)$$

- Solution

$$\Psi^{(1)} = -\frac{1}{2}\log^2(y_1 y_2 y_3) - 2\,\mathrm{Li}_2(1-y_1) - 2\,\mathrm{Li}_2(1-y_2) + \frac{\pi^2}{6} + \mathcal{O}(m^2)$$

$$\Psi^{(2)} = h(y_1 y_2 y_3) - 2\log(y_1 y_2 y_3)[f(y_1) + f(y_2)] + [g(y_1) + g(y_2)] + \mathcal{O}(m^2)$$

$$h(x) = -\frac{7}{40}\pi^4 + 2\zeta_3\log(x) - \frac{1}{4}\pi^2\log^2(x) - \frac{1}{24}\log^4(x)\,,$$

$$g(x) = -\frac{2}{3}\pi^2 H_2(x) + 2H_0(x)H_{2,0}(x) - 4H_{3,0}(x) - 2H_{2,0,0}(x) - 4H_{2,1,0}(x)\,,$$

$$f(x) = H_{0,1,0}(x)\,.$$

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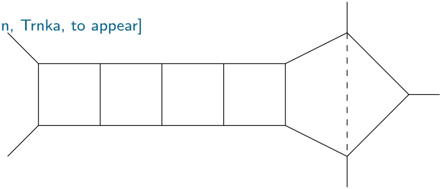

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$$f(x) = H_{0,1,0}(x)\,.$$

Summary

- Perspective: analytical results for planar $\mathcal{N} = 4$ SYM amplitudes at two loops and beyond
- Tools:
  - regulate infrared divergences by masses
  - momentum twistors to simplify integral basis
    also for other theories!
    especially powerful in $\mathcal{N} = 4$ SYM
    e.g. two-loop MHV amplitudes: only one integral topology
  - integrals satisfy second-order differential equations
  - differential equations have iterative structure in number of legs and loops
    analytic solutions as for Davydychev & Usyukina ladders!?
  - manifestation of underlying Yangian symmetry!?