Vanishing Chiral Algebras

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Introduction

The chiral algebras of 2d QFT with $(0, 2)$ SUSY:

- generalization of chiral rings of $(2, 2)$ theories
- ∞ dim
- $(0, 2)$ mirror sym, geometric Langlands, ...

The chiral algebras can vanish.

Implications:

- Supersymmetry is spontaneously broken
- A Fano manifold is covered by rational curves

Cohomology of $(0,2)$ SUSY

$(0,2)$ SUSY algebra:

$$Q_+^2 = \overline{Q}_+^2 = 0, \qquad \{Q_+, \overline{Q}_+\} = H - P.$$

Under the U(1) R-symmetry,

$$Q_+ : q = -1, \qquad \overline{Q}_+ : q = +1.$$

Write $Q = \overline{Q}_+$. The action of Q on $\mathcal{O}$ squares to 0,

$$[Q, [Q, \mathcal{O}\}\} = 0,$$

and increases q by 1.

Consider the Q-cohomology, graded by q.

Chiral Algebra

Q-cohomology classes vary holomorphically:

$$\partial_z \mathcal{O} = [H - P, \mathcal{O}] = [Q, [Q_+, \mathcal{O}\}\}.$$

If we define

$$[\mathcal{O}_i(z)] \cdot [\mathcal{O}_j(w)] = [\mathcal{O}_i(z)\mathcal{O}_j(w)],$$

they inherit OPEs from the underlying theory:

$$[\mathcal{O}_i(z)] \cdot [\mathcal{O}_j(w)] \sim c_{ij}{}^k(z-w)[\mathcal{O}_k(w)].$$

We obtain a chiral algebra, an operator product algebra of holomorphic fields.

The Model

A sigma model with right-moving fermions:

$$S = \int_\Sigma d^2z (g_{i\bar{j}} \partial_{\bar{z}} \phi^i \partial_z \phi^{\bar{j}} + i g_{i\bar{j}} \psi^i_+ D_z \bar{\psi}^{\bar{j}}_+) \,.$$

If the target space X is Kähler, it has $(0,2)$ SUSY:

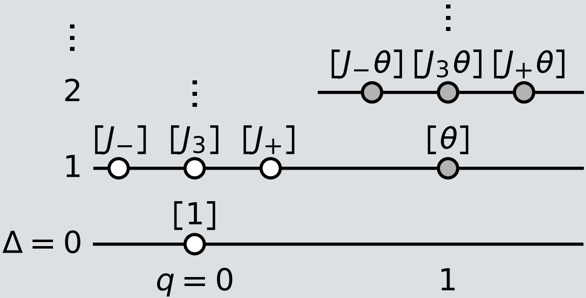
$$[Q, \phi^i] = 0, \qquad [Q, \phi^{\bar{\imath}}] = -i\bar{\psi}^{\bar{\imath}}_+,$$
$$\{Q, \psi^i_+\} = \partial_{\bar{z}} \phi^i, \quad \{Q, \bar{\psi}^{\bar{\imath}}_+\} = 0.$$

The other supercharge is $Q_+ = Q^\dagger$.

Witten studied the chiral algebra perturbatively.
[hep-th/0504078]

Perturbatively: $X = \mathbb{P}^1$

PCA for $X = \mathbb{P}^1$ looks like this:



Here $\theta = R_{i\bar{j}} \partial_z \phi^i \bar{\psi}^{\bar{j}}_+$. $\partial_z \theta$ does not appear at $\Delta = 2$:

$$[Q, T_{zz}] = \partial_z \theta.$$

If $c_1(X) \neq 0$, CA has no energy-momentum tensor.

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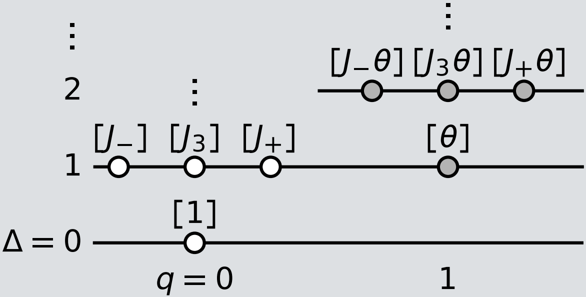


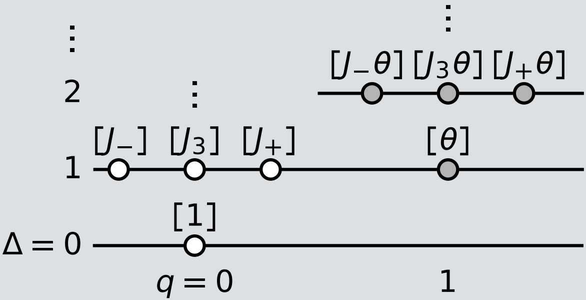
The Js generate the affine $\mathrm{SL}(2)$ algebra at $k = -2$:

$$J_- = g_{\phi\bar{\phi}}\partial_z\bar{\phi}, \quad J_3 = \phi g_{\phi\bar{\phi}}\partial_z\bar{\phi}, \quad J_+ = \phi^2 g_{\phi\bar{\phi}}\partial_z\bar{\phi}.$$

PCA for $X = G/B$ is a $\hat{\mathfrak{g}}$-module at the critical level.

Perturbatively: $X = \mathbb{P}^1$

PCA for $X = \mathbb{P}^1$ looks like this:



A Fock space structure:

$[1]$ & $[\theta]$: "ground states"; $q = 0$: "creation ops".

No state-operator isomorphism here!

Witten's Prediction

Witten thought:

Instantons "tunnel" between the "ground states,"

$$[1] \longleftrightarrow [\theta],$$

and induce

$$\{Q, \theta\} \sim 1.$$

Instantons violate q and Δ.

Then $\{Q, \mathcal{O}\theta\} \sim \mathcal{O}$, so instantons also pair up

$$[\mathcal{O}] \longleftrightarrow [\mathcal{O}\theta].$$

This explains the pattern.

Witten's Prediction

But $\{Q, \theta\} \sim 1$ means

$$1 = 0$$

in the Q-cohomology. Therefore,

$$[\mathcal{O}] = [1] \cdot [\mathcal{O}] = 0$$

and CA $= 0$. It vanishes.

Verified by an instanton computation.
[M.-C. Tan & JY, 0801.4782]

Extended to more general cases (e.g. $X = G/B$).
[JY, 1002.0028]

Even More Generally...

We claim:

The chiral algebra vanishes if $c_1(X) > 0$.

Proof (sketch):

1. Show CA is RG invariant.
2. In the IR we have a SCFT, so CA has an energy-momentum tensor.
3. Show CA has no such tensor in the UV.
4. The IR theory must be trivial and CA $= 0$.

Details to appear soon.

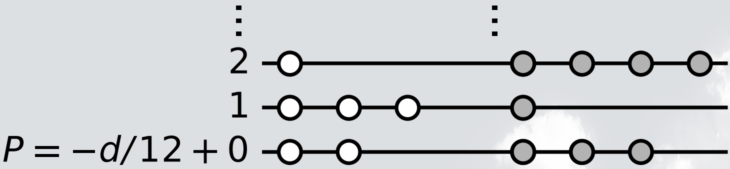
Supersymmetry Breaking

SUSY states are states annihilated by Q_+ and $\overline{Q}_+$.

Since $Q_+^\dagger = \overline{Q}_+$, they are states having

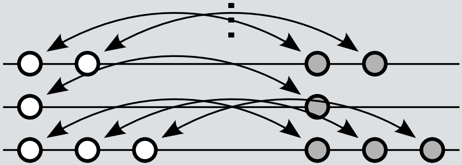
$$\{Q_+, \overline{Q}_+\} = H - P = 0.$$

There are ∞ of them, with $P = -d/12 + 0, 1, 2, \ldots$.

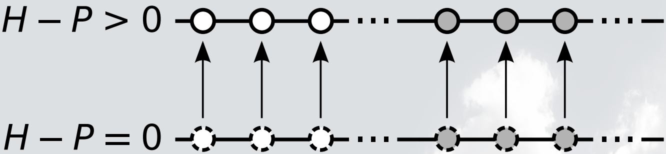


Supersymmetry Breaking

When SUSY is spontaneously broken, instantons pair up perturbative SUSY states



and lift all of them at once:



Supersymmetry Breaking

Consider the Q-cohomology of states.

This is a module over CA:

$$[\mathcal{O}] \cdot [|\Psi\rangle] = [\mathcal{O}|\Psi\rangle].$$

Suppose CA vanishes, $[1] = 0$. Then

$$[|\Psi\rangle] = [1] \cdot [|\Psi\rangle] = 0.$$

So the Q-cohomology of states vanishes as well.

This is isomorphic to the space of SUSY states.

Hence, there are no SUSY states.

Supersymmetry Breaking

The elliptic genus

$$\mathcal{E}(q) = \mathrm{Tr}\Big((-1)^F q^P\Big)$$

is the partition function on a torus. It counts

bosonic SUSY states − # fermionic SUSY states

at each energy level.

If $c_1(X) > 0$, then $\mathcal{E}(q) = 0$. This gives a “proof” of the Höhn–Stolz conjecture in the Kähler case:

$\mathcal{E}(q) = 0$ *if the target space admits a Riemannian metric with positive Ricci curvature.* (Stolz, 1996)

Rational Curves in Fano Manifolds

When CA $= 0$, a perturbative Q-cohomology class $[\mathcal{O}]$ exists such that instantons induce

$$\{Q, \mathcal{O}\} = 1.$$

Relevant instantons $C \colon \mathbb{P}^1 \to X$ have

$$\langle c_1(X), [C] \rangle = q + 1 \quad (q\text{: R-charge of } \mathcal{O}).$$

X must be covered by such Cs.

$q \leq d$, so we have a refinement of Mori's theorem:

Through every point in a Fano manifold there exists C such that $-K_X \cdot C \leq d + 1$. (Mori, 1979)

Summary

Suppose $c_1(X) > 0$. Perturbatively,

- CA is ∞-dim, has no energy-momentum tensor
- ∞ SUSY states

Nonperturbatively,

- $\mathrm{CA} = 0$
- SUSY is spontaneously broken

This result gives

- the Kähler case of the Höhn–Stolz conjecture
- a refinement of Mori's theorem