

Algebra^{*ic geometry*} for Quantum Fields

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The polylog as a Hodge structure

Iterated integrals: obvious Hopf algebra structure

$$\begin{pmatrix} 1 & 0 & 0 \\ -Li_1(z) & 2\pi i & 0 \\ -Li_2(z) & 2\pi i \ln(z) & (2\pi i)^2 \end{pmatrix} = (C_1, C_2, C_3) \quad (1)$$

$$\text{Var}(\Im Li_2(z) - \ln |z| \Im Li_1(z)) = 0 \quad (2)$$

Hodge structure from Hopf algebra structure: branch cut ambiguities columnwise

Griffith transversality \Leftrightarrow differential equation

Hopf algebra of graphs $H = \mathbb{Q}1 \oplus \bigoplus_{j=1}^{\infty} H^j$

► The coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \overbrace{\sum_{\gamma = \cup_i \gamma_i, \omega_4(\gamma_i) \geq 0}}^{\Delta'(\Gamma)} \gamma \otimes \Gamma/\gamma \quad (3)$$

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► The antipode

$$S(\Gamma) = -\Gamma - \sum S(\gamma)\Gamma/\gamma = -m(S \otimes P)\Delta \quad (4)$$

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$$G_V^H \ni \Phi \Leftrightarrow \Phi : H \rightarrow V, \Phi(h_1 \cup h_2) = \Phi(h_1)\Phi(h_2) \quad (5)$$

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$$\begin{aligned} S_R^\Phi(\Gamma) &= -R \left(\Phi(h) - \sum S_R^\Phi(\gamma)\Phi(\Gamma/\gamma) \right) \\ &= -R \Phi \left(m(S_R^\Phi \otimes \Phi P)\Delta(\Gamma) \right) \end{aligned} \quad (6)$$

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- ▶ The renormalized Feynman rules

$$\Phi_R = m(S_R^\Phi \otimes \Phi)\Delta \quad (7)$$

An Example

- ▶ The co-product

$$\Delta' \left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \\ \text{Diagram 6} \\ \text{Diagram 7} \end{array} \right) = 3 \begin{array}{c} \text{Diagram 8} \\ \otimes \\ \text{Diagram 9} \end{array} + 2 \begin{array}{c} \text{Diagram 10} \\ \otimes \\ \text{Diagram 11} \end{array} + \begin{array}{c} \text{Diagram 12} \\ \otimes \\ \text{Diagram 13} \end{array} .$$

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$$\Delta' \left(\text{[diagram: 7 vertices in a chain, last is a circle with a dot]} \right) = 3 \text{[diagram: vertex with a dot]} \otimes \text{[diagram: vertex with a dot]} \\ + 2 \text{[diagram: vertex with a semi-circle]} \otimes \text{[diagram: vertex with a dot]} + \text{[diagram: vertex with a circle]} \otimes \text{[diagram: vertex with a dot]} .$$

- ▶ The counterterm

$$S_R^\Phi \left(\text{[diagram: 7 vertices in a chain, last is a circle with a dot]} \right) = -Rm \left[S_R^\Phi \otimes \Phi P \right] \times \\ \times \Delta \left(\text{[diagram: 7 vertices in a chain, last is a circle with a dot]} \right) \\ = -R \left\{ \Phi \left(\text{[diagram: 7 vertices in a chain, last is a circle with a dot]} \right) + \right. \\ \left. + R \left[\Phi \left(3 \text{[diagram: vertex with a dot]} + 2 \text{[diagram: vertex with a semi-circle]} + \text{[diagram: vertex with a circle]} \right) \right] \Phi \left(\text{[diagram: vertex with a dot]} \right) \right\}$$

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$$\begin{aligned} S_R^\Phi \left(\text{diagram} \right) &= -Rm \left[S_R^\Phi \otimes \Phi P \right] \times \\ &\quad \times \Delta \left(\text{diagram} \right) \\ &= -R \left\{ \Phi \left(\text{diagram} \right) + \right. \\ &\quad \left. + R \left[\Phi \left(3 \text{diagram} + 2 \text{diagram} + \text{diagram} \right) \right] \Phi \left(\text{diagram} \right) \right\} \end{aligned}$$

- ▶ The renormalized result

$$\begin{aligned} \Phi_R &= (\text{id} - R)m(S_R^\Phi \otimes \Phi P)\Delta \left(\text{diagram} \right) \\ &= (\text{id} - R) \left\{ \Phi \left(\text{diagram} \right) + \right. \\ &\quad \left. + R \left[\Phi \left(3 \text{diagram} + 2 \text{diagram} + \text{diagram} \right) \right] \Phi \left(\text{diagram} \right) \right\} \end{aligned}$$

sub-Hopf algebras

- ▶ summing order by order

$$c_k^r = \sum_{|\Gamma|=k, \text{res}(\Gamma)=r} \frac{1}{|\text{Aut}(\Gamma)|} \Gamma \Rightarrow \Delta(c_k^r) = \sum_j \text{Pol}_j(c_m^s) \otimes c_{k-j}^r. \quad (8)$$

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- ▶ Hochschild closedness

$$X^r = 1 \pm \sum_j c_j^r \alpha^j = 1 \pm \sum_j \alpha^j B_+^{rj}(X^r Q^j(\alpha)), \quad (9)$$

$$Q^j = \frac{X^v}{\sqrt{\prod_{\text{edges } e \text{ at } v} X^e}}. \text{ Evaluates to invariant charge.}$$

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- ▶ $bB_+^{r;j} = 0$.

$$\Delta B_+^{r;j}(X) = B_+^{r;j}(X) \otimes 1 + (id \otimes B_+^{r;j})\Delta(X). \quad (10)$$

Implies locality of counterterms upon application of Feynman rules

$$\Phi B_+^{r;j}(X) = \int d\mu_{r;j} \Phi(X):$$

$$\bar{R}(\Gamma) = m(S_\Phi^R \otimes \Phi P) \Delta B_+^{r;j} = \int d\mu_{r;j} \Phi^R(X). \quad (11)$$

Symmetry

- ▶ Ward and Slavnov–Taylor ids

$$i_k := c_k^{\bar{\psi}\psi} + c_k^{\bar{\psi}A\psi} \quad (12)$$

span Hopf (co-)ideal I :

$$\Delta(I) \subseteq H \otimes I + I \otimes H. \quad (13)$$

$$\Delta(i_2) = i_2 \otimes 1 + 1 \otimes i_2 + (c_1^{\frac{1}{4}F^2} + c_1^{\bar{\psi}A\psi} + i_1) \otimes i_1 + i_1 \otimes c_1^{\bar{\psi}A\psi}.$$

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- ▶ Ideals for Slavnov–Taylor ids generated by equality of renormalized charges, also for the master equation in Batalin–Vilkovisky (see Walter van Suijlekom’s work)
- ▶ Similar ideals for the core Hopf algebra are respected by the BCFW recursion, and fit naturally with the structure of perturbative quantum gravity

BCFW and the core Hopf algebra

- ▶ a sequence of quotient Hopf algebras by looking at short distance singularities in $2k$ dimensions

$$H_0 \subset H_2 \subset H_4 \subset \dots \subset H_{2k} \subset \dots \subset H_\infty \quad (14)$$

$$\begin{aligned} \Delta'_4 \left(\text{diagram} \right) &= \text{diagram} \otimes \text{diagram} \\ \Delta'_\infty \left(\text{diagram} \right) &= 2 \text{diagram} \otimes \text{diagram} + \text{diagram} \otimes \text{diagram} \end{aligned} \quad (15)$$

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$$H_0 \subset H_2 \subset H_4 \subset \dots \subset H_{2k} \subset \dots \subset H_\infty \quad (14)$$

$$\begin{aligned} \Delta'_4 \left(\text{triangle with two internal lines} \right) &= \text{loop with two external lines} \otimes \text{fish diagram} \\ \Delta'_\infty \left(\text{triangle with two internal lines} \right) &= 2 \text{ (circle with two external lines)} \otimes \text{ (circle with two external lines)} + \text{ (loop with two external lines)} \otimes \text{ (fish diagram)} \end{aligned} \quad (15)$$

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$$\begin{aligned} \Delta'_4 \left(\text{triangle with lens} \right) &= \text{crossing} \otimes \text{fish} \\ \Delta'_\infty \left(\text{triangle with lens} \right) &= 2 \left(\text{circle with legs} \right) \otimes \left(\text{circle with legs} \right) + \text{crossing} \otimes \text{fish} \end{aligned} \quad (15)$$

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- ▶ Hochschild cohomology, co-ideals: trade loop for leg expansion

The diagram shows an equation between three vertices. On the left is a single vertex with n legs, labeled with $1, \dots, n, \dots$. This is equal to a sum over j of two vertices. The first vertex in the sum has n legs, with the j -th leg labeled $j+1$ and the $(j+1)$ -th leg labeled $1-z_j$. The second vertex in the sum has n legs, with the j -th leg labeled j and the $(j+1)$ -th leg labeled $2+z_j$. The two vertices in the sum are connected by a horizontal line with a $-+$ label above and a $+ -$ label below.

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$$\begin{aligned} \Delta'_4 \left(\text{fish diagram} \right) &= \text{self-energy} \otimes \text{fish diagram} \\ \Delta'_\infty \left(\text{fish diagram} \right) &= 2 \left(\text{loop with external legs} \right) \otimes \text{fish diagram} + \text{self-energy} \otimes \text{fish diagram} \end{aligned} \quad (15)$$

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The diagram shows an equation between three circular diagrams. On the left is a circle with n external legs labeled 1, ..., n. This is equal to a sum over j of two terms. The first term is a circle with n external legs labeled 1, ..., j+1. The second term is a circle with n external legs labeled 1-z_j, ..., j. A horizontal line connects the two circles in the sum, with a '+' sign above and a '-' sign below it. The rightmost circle has n external legs labeled 2+z_j, 3, ..., j.

- ▶ KLT relations or kinematic STU \leftrightarrow co-ideal respected

Kinematics and Cohomology

- ▶ Exact co-cycles

$$[B_+^{rj}] = B_+^{rj} + b\phi^{rj} \quad (16)$$

with $\phi^{rj} : H \rightarrow \mathbb{C}$

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- ▶ Variation of momenta

$$G^R(\{g\}, \ln s, \{\Theta\}) = 1 \pm \Phi_{\ln s, \{\Theta\}}^R(X^r(\{g\})) \quad (17)$$

with $X^r = 1 \pm \sum_j g^j B_+^{rj}(X^r Q^j(g))$, $bB_+^{rj} = 0$. Also,

$$G^r = \left[\sum_{j=1}^{\infty} \gamma_j(\{g\}, \{\Theta\}) \ln^j s \right] + \overbrace{G_0^r}^{\text{abelian factor}} \quad (18)$$

Then, for MOM and similar schemes (not MS!):

$$\{\Theta\} \rightarrow \{\Theta'\} \Leftrightarrow B_+^{rj} \rightarrow B_+^{rj} + b\phi^{rj}.$$

$$\Phi_{L_1+L_2, \{\Theta\}}^R = \Phi_{L_1, \{\Theta\}}^R \star \Phi_{L_2, \{\Theta\}}^R.$$

Leading log expansions and the RGE

- ▶ The invariant charge Q^v
For each vertex v , a charge Q^v :

$$Q^v(g) = \frac{X^v(g)}{\prod_e \sqrt{X^e}}, \quad (19)$$

e adjacent to v .

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$$\left(\partial_L + \beta(g) \partial_g - \sum_{e \text{ adj } r} \gamma_1^e \right) G^r(g, L) = 0 \quad (20)$$

rewrites in terms of the Dykin operator

($\gamma_1^r(g) = S \star Y(X^r(g))$):

$$\gamma_k^r(g) = \frac{1}{k} \left(\gamma_1^r(g) - \sum_{j \in R} s_j \gamma_1^j g \partial_g \right) \gamma_{k-1}^r(g) \quad (21)$$

Ordinary differential equations vs DSE

- ▶ RGE+DSE
the iterated integral structure

$$\Phi^R(B_+^{r;j}(X)) = \int \Phi^R(X) d\mu_{r;j} \quad (22)$$

allows to combine $X^r = 1 \pm \sum_j B_+(X^r Q^j)$ with RGE to

$$\gamma_1^r = P(g) - [\gamma_1^r(g)]^2 + \sum_{j \in R} s_j \gamma_1^j g \partial_g \gamma_1^r(g). \quad (23)$$

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- ▶ massless gauge theories
 $\beta(g) = g\gamma_1(g)/2$ for γ_1 anomalous dim of gauge propagator

$$\gamma_1(g) = \overbrace{P(g)}^{\text{existence assumed}} - \gamma_1(g)(1 - g\partial_g)\gamma_1(g) \quad (24)$$

(Ward Id QED, background field gauge (Abbott) QCD)

Limiting mixed Hodge structures

- ▶ Hopf algebra from flags

$$f := \gamma_1 \subset \gamma_2 \subset \dots \subset \Gamma, \Delta'(\gamma_{i+1}/\gamma_i) = 0 \quad (25)$$

The set of all such flags $F_\Gamma \ni f$ determines Hopf algebra structure, $|F_\Gamma|$ is the length of the flag.

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- ▶ It also determines a column vector $v = v(F_\Gamma)$ and a nilpotent matrix $(N) = (N(|F_\Gamma|))$, $(N)^{k+1} = 0$, $k = \text{corad}(\Gamma)$ such that

$$\lim_{t \rightarrow 0} (e^{-\ln t(N)}) \Phi_R(v(F_\Gamma)) = (c_1^\Gamma(\Theta) \ln s, c_2^\Gamma(\Theta), c_k^\Gamma(\Theta) \ln^k s)^T \quad (26)$$

where k is determined from the co-radical filtration and t is a regulator say for the lower boundary in the parametric representation.

The Feynman graph as a Hodge structure

Hopf algebra structure as above

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ \text{Diagram 1} & \text{Diagram 2} & 0 & 0 & 0 \\ \text{Diagram 3} & 0 & \text{Diagram 4} & 0 & 0 \\ \text{Diagram 5} & 0 & 0 & \text{Diagram 6} & 0 \\ \text{Diagram 7} & \text{Diagram 8} & \text{Diagram 9} & \text{Diagram 10} & \text{Diagram 11} \end{array} \right) = (C_1, C_2, C_3, C_4, C_5)$$

$$\text{Var} \left(\mathfrak{S} \cdot \text{Diagram 7} - \left[\mathfrak{R} \cdot \text{Diagram 8} \cdot \mathfrak{S} \cdot \text{Diagram 6} \right] + \dots \right) = 0$$

Hodge structure: cut-reconstructability: from Hopf algebra structure:
branch cut ambiguities columnwise

Griffith transversality \Leftrightarrow differential equation? (Henn's talk!)

$$\zeta(s_1, \dots, s_k) = \sum_{n_i < n_{i+1}} \frac{1}{n_1^{s_1} \dots n_k^{s_k}}$$

► counting over \mathbb{Q}

$$1 - \frac{x^3 y}{1 - x^2} + \frac{x^{12} y^2 (1 - y^2)}{(1 - x^4)(1 - x^6)} = \prod_{n \geq 3} \prod_{k \geq 1} (1 - x^n y^k)^{D_{n,k}} \quad (27)$$

→ first irreducible MZV from planar graphs at 7 loops in scalar field theory (integrability???)

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- ▶ When is a graph reducible to MZVs? Francis Brown: when it has vertex width three.

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- ▶ When is a graph reducible to MZVs? Francis Brown: when it has vertex width three.
- ▶ Caution! Non-MZVs at eight loops from non-planar graphs, at nine loops from planar graphs ('A K_3 in ϕ^4 ', Brown and Schnetz). Proof from counting points $[X_\Gamma]$ on graph hypersurfaces X_Γ over \mathbb{F}_q , defined by vanishing of the first Symanzik polynomial. If the graph gives a MZV, $[X_\Gamma]$ better is polynomial in the prime power $q = p^n$. Alas, it is not in general, with counterexamples relating graphs to elliptic curves with complex multiplication, and point-counting function a modular form.

QED

- ▶ sub Hopf algebra for vacuum polarization suffices

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$P(x)$ twice

differentiable

$$\gamma_1(x_0) = \gamma_0 > 0$$

different solutions

distinguished by $e^{-\frac{1}{x}}$

behaviour

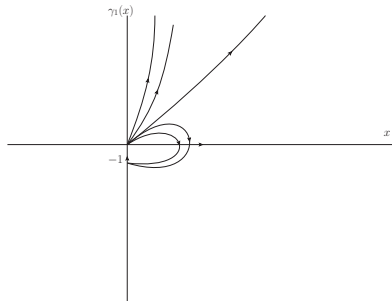
$$\frac{d\gamma_1}{dx} = \gamma_1 - \gamma_1^2 - P,$$

$$\frac{dx}{dL} = x\gamma_1$$

$$L = \int_{x_0}^{x(L)} \frac{dz}{z\gamma_1(z)}$$

- ▶ **separatrix exists and might have no Landau pole:**

$$D(P) = \int_{x_0}^{\infty} \frac{P(z)dz}{z^3} < \infty, \int_{x_0}^{\infty} \frac{2dz}{z\sqrt{1+4P(z)-1}} < \infty$$



QCD

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- ▶ $\gamma_1(g) = P(g) - \gamma_1(g)^2 + \gamma_1(g)g\partial_g\gamma_1(g)$ with $P(g) < 0$

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$P(g)$ twice differentiable
and concave near 0

unique solution which

flows into $(0, 0)$ at large Q^2

$$L = \int_{g_0}^{g(L)} \frac{dz}{z\gamma_1(z)} \rightarrow$$

$$L_\Lambda = - \int_{g(L_\Lambda)}^{\infty} \frac{dz}{z\gamma_1(z)},$$

$$L_\Lambda = \ln Q^2 / \Lambda_{QCD}$$

$$f_{disp}(Q^2) = \int_0^\infty \frac{\Im(f(\sigma))d\sigma}{\sigma + Q^2 - i\eta}$$

and ODE

- ▶ separatrix exists and gives asymptotic free solution, with finite mass gap for inverse propagator iff $\gamma_1(x) < -1$ for some $x > 0$.

$|D(P)| < \infty \rightarrow \gamma_1(x) \sim sx, x \rightarrow \infty$. That allows for dispersive methods as introduced by Shirkov et.al. in field theory.

