Some new methods in conformal QFT in four dimensions

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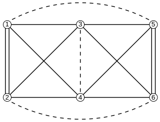


3 Topics

1. Perturbation theory without conformal anomaly
2. Chiral supersymmetric correlation functions
3. Structure of conformal correlation functions

ANOMALY-FREE PERTURBATION THEORY

Joint work with Michael Dütsch

AdS-CFT

Bulk fields and boundary fields

If Φ is a scalar AdS-covariant Wightman field on AdS_{D+1}, then its boundary limit

$$\phi(x) := \lim_{z \searrow 0} z^{-d} \Phi(x; z)$$

is a conformally covariant field on Minkowski spacetime $\mathbb{R}^D$ with scaling dimension d, **provided the limit exists** in the sense of correlation functions. [Bertola-Bros-Moschella-Schaeffer]

For Klein-Gordon fields on AdS, the boundary limit exists, and gives a generalized free field ("unparticle", Källen-Lehmann decomposition with continuous mass) with scaling dimension

$$d = \frac{1}{2}\Big(D + \sqrt{D^2 + 4M^2}\Big)$$

(e.g., [Fronsdal], [Witten]).

AdS-CFT

Idea:

Perturbation of the AdS-field $\Phi(X)$ ($X \equiv (z, x)$) results in a perturbation of the boundary field $\varphi(x)$.

Interacting conformal fields with anomalous dimensions are "closer" to a generalized free field of the same dimension, than to the massless free field of canonical dimension.

If the bulk renormalization is AdS-invariant, then the boundary field is conformally covariant without a conformal anomaly, **provided the limit exists** for the perturbed field.

The existence of the limit has to be studied **after renormalization**.

Causal perturbation theory

- x-space formalism best suited for curved spacetime.
- Locality is manifest in every order.
- Expansion formula

$$\phi_{g\mathcal{L}} = \sum \frac{(\hbar/i)^n}{n!} R((g\mathcal{L})^{\otimes n}; \phi)$$

- Operator valued distributions $g^{\otimes n} \mapsto R(g\mathcal{L}^{\otimes n}; \phi)$ are of the form (numerical distributions r) $\times$ (Wick products)
- Retarded distributions r are ill-defined at $x_i = x_j$. By locality, they extend uniquely to coinciding points, **except to the "total diagonal"** $x_1 = \cdots = x_n = x$.

Renormalization = extension of distributions
to points where they are ill-defined.

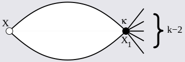
Boundary approach

Some issues of the "boundary approach":

- The free bulk field can be represented "holographically" as a family of boundary GFFs.
- Renormalized perturbation theory "around generalized free fields" is not an established theory, because of the enormous renormalization freedom (the Borchers class of a GFF is much bigger than the Wick products).
- The standard renormalization recipes, applied to the Källen-Lehmann decomposition of the boundary GFFs, do not result in a renormalization of the bulk field: Boundary renormalization does not respect **bulk locality**.
- Moreover, renormalization on the boundary in general breaks conformal symmetry through a renormalization scale μ^2.

Bulk approach

Some issues of the "bulk approach":

- AdS-invariant renormalization in the bulk is "always" possible.
- Provided the boundary limit exists, unbroken conformal symmetry is ensured.
- 

The boundary limit of renormalized composite fields, such as $:\varphi^2:$ with ϕ^4 interaction, requires convergence of renormalized expressions like

$$\lim_{z \searrow 0} z^{-2d} \cdot \int \frac{dz_1}{z_1^{D+1}}\, \gamma(z_1) \int d^D x_1\, g(x_1)\, r_{\mathrm{fish}}(X_1; X) {:}\phi^2(X_1){:}$$

where r_{fish} is a renormalization of the product of distributions

$$\left(\Delta^+_{\mathrm{AdS}}(X_1, X)^2 - \Delta^+_{\mathrm{AdS}}(X, X_1)^2\right) \theta(x^0 - x_1^0).$$

Cancellations

- The z-dependence of the integral cannot be read off the z-dependence of the integrand.
- An apriori estimate of the integral goes only $\sim z$. The boundary limit requires $\sim z^{2d}$. Hence, for typical dimensions $d > 1$, nontrivial **cancellations must occur**.

Example:

AdS dimension $D + 1 = 4$, AdS mass $M = 0$, hence scaling dimension $d = 3$.

- Task: Analyze the behaviour of the integral at small z. The five leading powers must be exactly zero (in the "partial adiabatic limit" where $\gamma(0) = 1$). Logarithms?

Mechanism of cancellations

Differential renormalization:

$$r_{\rm fish}(X_1;X) = \Box_{X_1}\left[\delta F(v)\theta(x^0 - x_1^0)\right]$$

where $v = \frac{z^2+z_1^2-(x-x_1)^2}{2zz_1}$ is the AdS distance, and $\delta F(v)$ is the discontinuity of an analytic function with a cut along $[-1,1]$ (corresponding to timelike separation in the bulk).
The integral becomes

$$\int \frac{dz_1}{z_1^{D+1}} \int d^D x_1\; \delta F(v)\, \theta(x^0 - x_1^0)\, \Box_{X_1}\left[\gamma(z_1) g(x_1) :\!\phi^2(X_1)\!:\right].$$

- First perform the "inner" integral at fixed (timelike) bulk distance v, and then the "outer" integral over v involving the renormalized distribution (through $\delta F(v)$).

Mechanism of cancellations

- The inner integral $I(z,v)$ admits a power series expansion $j_n(v)z^n$, with a $\log z$-term first appearing in order z^6.
- The subsequent (distributional) outer integral $\int_{-1}^{+1} \delta F(v) I(z,v)$ precisely annihilates the five leading orders $j_n(v)$ $(n = 1, \ldots, 5)$.
- The $z^6 \log z$-term is **local**, i.e., it multiplies $g(x){:}\varphi^2{:}(x)$. This permits to interpret it as the leading term of an anomalous dimension.
- $d_{\varphi^2} = 6 - \frac{7g}{6\pi^2} + \mathcal{O}(g^2)$ for massless ϕ^4 in $D + 1 = 4$.

Resumé

- The cancellation relies on the detailed knowledge of the transcendental discontinuity δF in the "outer integral", where F arises in the distributive extension of products like $\left(\Delta^{+}_{\mathrm{AdS}}(X, X_1)\right)^2$ by differential renormalization. In the simplest case $(D = 3, M = 0)$, this involves Li_2 and Li_3.
- The mechanism seems to reveal a **hidden symmetry**.
- The general features for the "inner integral" are systematic, i.e., independent of the special values $D = 3,\ M = 0$.
- The generality of existence of the limit requires and deserves further exploration.

SUPERSYMMETRIC CORRELATION FUNCTIONS

PhD thesis Holger Knuth

PhD thesis H. Knuth

Superfields $\Phi(x,\theta) = \phi(x) + \bar{\theta}\psi(x) + \dots$
$N = 1$ chiral superfields $D\Phi(x,\theta) = 0$.

Structure of chiral superscalar correlation functions

16 conformal 4-point invariants, but

$$\langle \Phi\Phi\bar{\Phi}\bar{\Phi} \rangle = e^{\mathcal{D}_{\mathrm{Ward}}} f(I_1, I_2).$$

- I_1, I_2 reduce to the ordinary conformal cross-ratios.
- $\mathcal{D}_{\mathrm{Ward}}$ is a **universal nilpotent differential operator** (involving the nilpotent invariants).
- Knowing the correlation of the "leading" conformal fields $\phi(x)$ completely determines the correlations of the entire chiral super multiplet.

This last fact has been used before ("Ward identities"), but the implementation by a universal operator $\mathcal{D}$ was not known.

STRUCTURE OF CORRELATION FUNCTIONS IN 4D CFT

Joint work with N. Nikolov, I. Todorov (and several students)

Cross ratios

Well known: Conformal symmetry implies

$$\langle \Omega, \phi(x_1) \ldots \phi(x_n) \Omega \rangle = \prod_{ij} (x_{ij}^2)^{\mu_{ij}} \times f(\text{cross ratios}).$$

Question:

Further restrictions on f(cross ratios)?

Hilbert space positivity

Hilbert space positivity

- **Unitarity bound** on representations of conformal group restricts quantum numbers $\lambda = (d, j_1, j_2)$ [Mack]: generically

$$d \geq j_1 + j_2 + 2.$$

- UB also holds for all fields in the OPE $\Rightarrow$ **pole bounds** [Nikolov-Todorov]
- $\| \sum \phi \ldots \phi \Omega \rangle \|^2 \geq 0 \Rightarrow$ **positivity and Cauchy-Schwarz inequalities** for PWE coefficients [Rühl-Lang, . . . , Nikolov-KHR-Todorov].

To compute the PWE coefficients, one needs to know the partial waves: closed form for 4-point PW [Dolan-Osborn] available.

Global Conformal Symmetry

Simplification:

Huygens' Principle

Equivalent are [Nikolov-Todorov]:

- Commutativity at timelike distance
- Laurent rationality of correlation functions
- True representation of conformal group

Natural **suspicion**: only free fields and their Wick products could comply with this principle

Indications that this might not be true!

First steps

- Huygens' Principle plus pole bounds imply that there are only finitely many possible "structures" for every n-point function.
- Closed expansion formula of rational 4-point structures into partial waves.
- Insight (from 4-point functions): positivity of PWE coefficients is quite restrictive at $j \to \infty$ (large-spin partial waves); certain structures can be ruled out by positivity.
- **Theorem** for $d = 2$ scalar fields [Nikolov-KHR-Todorov]: Positivity implies $\phi = :\varphi^2:$.

Beyond 4 points

- Twist = dimension minus spin = $d - j_1 - j_2$.
- Twist 2 $\Leftrightarrow$ conserved tensor fields
- $\Rightarrow$ twist-2 contribution to PWE is "biharmonic":

$$W = ((x-y)^2)^{d-1} \langle \phi \dots \phi \Pi_{\mathrm{twist2}} \phi(x) \phi(y) \rangle$$

solves $\Box_x W = \Box_y W = 0$.

- Either equation fixes the full twist-2 contribution, once the leading term $\sim ((x-y)^2)^{1-d}$ is known

Overdetermined problem $\Rightarrow$ **not every leading structure is admissible**. Severely constrains the pole structure.

Beyond 4 points

- New restrictions become effective only at $n \geq 6$ points.
- Admissible $n \geq 6$-point structures can exhibit "double poles" $\Rightarrow$ **not representable by free fields**. Higher than double poles are excluded.
- Classification of twist-2 leading terms with double poles [Bischoff].
- Classification of full twist-2 structures for six $d = 3$ fields [Rathlev, KHR].

Example:

Twist-2 6-point structure ($d = 3$) with double poles

Leading term:

$$u_0 = \left[\frac{\frac{1}{2}(15)(26)(34) - (15)(23)(46) - (15)(24)(36)}{(13)(14)(23)(24)\cdot(34)\cdot(35)(36)(45)(46)} \right]_{[1,2],[5,6]} .$$

Full twist-2 structure

$$u_0 \cdot g(t,s) g(t',s') + \left[\frac{(13)(24)\cdot(35)(46)}{\cdots\cdot(34)^2\cdot\ldots} \right]_{[1,2],[5,6]} \cdot (1 - g(t,s)g(t',s')),$$

where $s = uv = \frac{(12)(34)}{(13)(24)}$, $t = (1-u)(1-v) = \frac{(14)(23)}{(13)(24)}$, etc, and

$$g(s,t) = \int_0^1 dx \left[\frac{1-(1-t)x}{1-(1-t-s)x+sx^2} \right]^2 = 1 + \sum_{a,b\geq 1} \frac{2ab\cdot u^a v^b}{(a+b-1)_3} .$$

Work in progress:

Are non-free double pole structures in **conflict with positivity?**

Need to explore 6-point PWE coefficients, without knowing the 6-point partial waves!
Idea: use "projecting kernels" to reduce 6-point structures to 4-point structures:

$$\langle \phi\phi\phi\phi\phi\phi \rangle \to \langle \phi_\lambda \phi\phi \phi_\lambda \rangle .$$

Then use 4-point methods.

(Outcome unknown)

THANK YOU